Semester	**IV**	**Corse Title**	**Engineering Mathematics-IV**	**Course Code**	**18MAT-41**
Teaching Period	**50 Hours**	**L – T – P – TL**	**2 - 1 - 0 - 3**	**SEE**	**3 Hours**
CIE	**40 Marks**	**SEE**	**60 Marks**	**Total**	**100 Marks**
CREDITS - 03					

Course objectives:

- To provide an insight into applications of complex variables, conformal mapping and special functions arising in potential theory, quantum mechanics, heat conduction and field theory.
- To develop probability distribution of discrete, continuous random variables and joint probability distribution occurring in digital signal processing, design engineering and microwave engineering.

:: Module-1 :: (10 Hours)

Calculus of complex functions: Review of function of a complex variable, limits, continuity, and differentiability. Analytic functions: Cauchy-Riemann equations in Cartesian and polar forms and consequences. Construction of analytic functions (Milne Thomson method problems).

RBTL- L1, L2

:: Module-2 :: (10 Hours)

Conformal transformations: Introduction. Discussion of transformations: $w = Z^2$, $w = e^z$, $w = z + \frac{1}{z}$, $z \neq 0$. Bilinear Transformations- Problems.

Complex integration: Line integral of a complex function-Cauchy’s theorem and Cauchy’s integral formula and problems.

RBTL-L1, L2

:: Module-3 :: (10 Hours)

Probability Distributions: Basic concepts of probability theory. Random variables (discrete and continuous), probability mass/density functions. Binomial, Poisson, exponential and normal distributions and problems.

RBTL-L1, L3

:: Module-4 :: (10 Hours)

Statistical Methods: Correlation and regression-Karl Pearson’s coefficient of correlation -problems. Regression analysis-lines of regression –problems.

Curve Fitting: Curve fitting by the method of least squares- fitting the curves of the form - $y = ax + b, y = ax^b$ *and* $y = ax^2 + bx + c$.

RBTL-L2, L3

:: Module-5 :: :(10 Hours)

Joint probability distribution: Joint Probability distribution for two discrete random variables, expectation and covariance.

Sampling Theory: Introduction to sampling distributions, standard error, Type-I and Type-II errors. Test of hypothesis for means. Stochastic process: Stochastic processes, probability vector, stochastic matrices, fixed points, regular stochastic matrices, Markov chains, higher transition probability – simple problems.

RBTL-L2, L3, L4

L1- Remembering, L2- Understanding, L3-Applying, L4-Analyzing.

Course outcomes:

At the end of the course the student will be able to:

- CO1: Use the concepts of analytic function and complex potentials to solve the problems arising in electromagnetic field theory.
- CO2: make use of conformal transformation and complex integral arising in aerofoil theory, fluid flow visualization and image processing.
- CO3: Apply discrete and continuous probability distributions in analyzing the probability models arising in engineering applications.
- CO4: Make use of the correlation and regression analysis to fit a suitable mathematical model for the statistical data.
- CO5: Construct joint probability distributions and demonstrate the validity of testing the hypothesis.

Question paper pattern:

- The question paper will have ten full questions carrying equal marks.
- Each full question will be for 20 marks.
- There will be two full questions (with a maximum of three sub- questions) from each module.
- Each full question will have sub- question covering all the topics under a module.

The students will have to answer five full questions, selecting one full question from each module.

Textbooks

1. Advanced Engineering Mathematics, E. Kreyszig, John Wiley & Sons, 10th Edition, 2016.
2. Higher Engineering Mathematics, B.S. Grewal, Khanna Publishers, 44th Edition, 2017.

Reference Books

1. Higher Engineering Mathematics, B.V.Ramana, McGraw Hill, 11th Edition, 2010.
2. A Text Book of Engineering Mathematics, N.P.Bali and Manish Goyal, Laxmi Publications, 2014.

module - 01

Calculus of Complex functions I

A number of the form $Z = x + iy$ where $x, y \in R$ is called a complex number.

x is called real part of Z
y is called imaginary part of Z
$i = \sqrt{-1}$ and $i^2 = -1$

If $Z = x + iy$ is a complex number then $\bar{Z} = x - iy$ is called conjugate of complex number.

NOTE①:

① $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

② $e^{ix} = \cos x + i\sin x$

③ $e^{-ix} = \cos x - i\sin x$

④ $\sin ix = i\sinh x$

⑤ $\cos ix = \cosh x$

NOTE②:

Demoiver's Theorem:-

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
where n is a real number

Function of a Complex Variable, Limit, Continuity and differentiability

Function of a complex Variable

$w = f(z)$ is called a function of Complex variable z. w is said to be single valued or many valued function of z there Correspands one or morethan one value of w.

Since $z = x + iy$ (or) $z = re^{i\theta}$

$w = f(z) = u(x, y) + iv(x, y)$ [Cartegian form]

$w = f(z) = u(r, \theta) + iv(r, \theta)$ [polar form]

Limit: A complex valued function $f(z)$ is defined in the neighbourhood of a point z_0 is said to have a limit l as $z \to z_0$, if for every $\varepsilon > 0$ however small ∃ a +ve real number δ ∋; $|f(z) - l| < \varepsilon$ when $|z - z_0| < \delta$,

This is written as $\lim_{z \to z_0} f(z) = l$

Continuity: A complex valued function $f(z)$ is said to be Contineous at $z = z_0$, if $f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$

i.e $|f(z) - f(z_0)| < \varepsilon$ when $|z - z_0| < \delta$

Differentiability: A complex valued function $f(z)$ is said to be differentiable at $z = z_0$, if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique.

Suppose we write $\delta z = z - z_0$ then $z \to z_0$, $\delta z \to 0$ hence

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(\delta z + z_0) - f(z_0)}{\delta z}$$

Analytic function and Connected theorems

A Complex Valued function $w=f(z)$ is said to be analytic at a point $z=z_0$ if $\frac{dw}{dz}=f'(z)=\underset{\delta z\to 0}{\mathrm{lt}}\frac{f(z+\delta z)-f(z}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0.

Theorem ①:-

[Cauchy-Riemann equations in the Cartesian form]

Statement: The necessary conditions that the function $w=f(z)=u(x,y)+iv(x,y$ may be analytic at any point $z=x+iy$ is that ∃ four Contineous first order partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and satisfy the equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

(or) $u_x=v_y$ and $v_x=-u_y$

These are known as C-R equations.

Proof: Let $f(z)$ be analytic at the point $z=x+iy$ and hence by definition,

$$f'(z)=\underset{\delta z\to 0}{\mathrm{lt}}\frac{f(z+\delta z)-f(z)}{\delta z}$$ exists & unique.

In Cartesian form

$f(z)=u(x,y)+iv(x,y)$.

and δz is the increment in Z corresponding to the increments δx and δy in x & y.

$$f(Z) = u(x,y) + iv(x,y)$$

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

w.K.T $f'(z) = \underset{\delta z \to 0}{\text{lt}} \dfrac{f(z+\delta z) - f(z)}{\delta z}$

$$f'(z) = \underset{\delta z \to 0}{\text{lt}} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - [u(x,y) + iv(x,y)]]}{\delta z}$$

$$f'(z) = \underset{\delta z \to 0}{\text{lt}} \frac{[u(x+\delta x, y+\delta y) - u(x,y)] + i\{v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{lt}} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \underset{\delta z \to 0}{\text{lt}} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \quad — ①$$

Now $\delta z = Z + \delta z - Z$ where $Z = x+iy$

$$\delta z = x+iy + \delta x + i\delta y - (x+iy)$$

$$\delta z = \cancel{x}+\cancel{iy} + \delta x + i\delta y - \cancel{x} - \cancel{iy}$$

$$\delta z = \delta x + i\delta y$$

Since δz tends to zero we have two cases

Case ①: let $\delta y = 0$ so that $\delta z = \delta x$ and $\delta z \to 0$ then $\delta x \to 0$

equation ① becomes

$$f'(z) = \underset{\delta x \to 0}{\text{lt}} \frac{[u(x+\delta x, y) - u(x,y)]}{\delta x} + i \underset{\delta x \to 0}{\text{lt}} \frac{[v(x+\delta x, y) - v(x,y)]}{\delta x}$$

These limits form the basic definition are the partial derivatives of u & v w.r.to x.

$$\boxed{\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \text{ — ②}$$

Case②: let $\delta x = 0$ so that $\delta z = i\delta y$ and $\delta z \to 0$ then $i\delta y \to 0$ or $\delta y \to 0$ equation ① becomes

$$f'(z) = \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) - u(x,y)]}{i\delta y} + i \lim_{\delta y \to 0} \frac{[v(x, y+\delta y) - v(x,y)]}{i\delta y}$$

NOTE: [i is in the denominator
∴ we need to rationalize]
$\frac{1}{i} = -i$

$$= -i \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) - u(x,y)]}{\delta y} + \lim_{\delta y \to 0} \frac{[v(x, y+\delta y) - v(x,y)]}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \text{ — ②}$$

equate the RHS of ② and ③

$$\frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = \frac{\partial v}{\partial y} - \frac{i\partial u}{\partial y}$$

equating the real part and imaginary part

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

or

$u_x = v_y$ and $v_x = -u_y$

Thye are Known as C-R equations

~~Q~~ Theorem - 2 :-

[Cauchy Riemann equation in polar form]

Statement :- If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ & satisfy the equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(or) $ru_r = v_\theta$ and $rv_r = -u_\theta$

These are known as Cauchy-Riemann equations in the polar form.

Proof :- Let $f(z)$ be analytic at a point $z = re^{i\theta}$ and hence by definition

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists and unique

In the polar form $f(z) = u(r,\theta) + iv(r,\theta)$ & let δz be the increment in z corresponding to the increments δr, $\delta\theta$ in r, θ.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) - u(r,\theta)]}{\delta z} + i[V(r+\delta r, \theta+\delta\theta) - V(r,\theta)]$$

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) - u(r,\theta)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[V(r+\delta r, \theta+\delta\theta) - V(r,\theta)]}{\delta z} \quad \text{—} ①$$

consider $z = re^{i\theta}$

Since z is a function of two variables r, θ we have

$$\delta z = \frac{\partial z}{\partial r}\cdot\delta r + \frac{\partial z}{\partial \theta}\cdot\delta\theta$$

$$= \frac{\partial}{\partial r}(re^{i\theta})\,\delta r + \frac{\partial}{\partial\theta}(re^{i\theta})\,\delta\theta$$

$$= e^{i\theta}\cdot\frac{\partial}{\partial r}(r)\cdot\delta r + r\frac{\partial}{\partial\theta}(e^{i\theta})\,\delta\theta$$

$$= e^{i\theta}\cdot(1)\,\delta r + r e^{i\theta}\cdot\frac{\partial}{\partial\theta}(i\theta)\cdot\delta\theta$$

$$= e^{i\theta}\cdot\delta r + r e^{i\theta}\cdot i\cdot\delta\theta$$

$$\delta z = e^{i\theta}\cdot\delta r + i r e^{i\theta}\,\delta\theta$$

Since δz tends to zero we have 2 cases

Case ①: let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\cdot\delta r$

and $\delta z \to 0$ then $\delta r \to 0$

Now equation ① becomes

$$f'(z) = \lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) - u(r,\theta)]}{e^{i\theta}\,\delta r} + i \lim_{\delta r \to 0} \frac{[V(r+\delta r, \theta) - V(r,\theta)]}{e^{i\theta}\cdot\delta r}$$

$$= \frac{1}{e^{i\theta}} \left\{ \lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) - u(r,\theta)]}{\delta r} + i \lim_{\delta r \to 0} \frac{[v(r+\delta r, \theta) - v(r,\theta)]}{\delta r} \right\}$$

$$f'(z) = e^{-i\theta} \left[\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right] \text{———} ②$$

Case ②: Let $\delta r = 0$ so that $\delta z = r i e^{i\theta} \delta\theta$

$\delta z \to 0$ then $\delta\theta \to 0$

Now eqn ① becomes

$$f'(z) = \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{i r e^{i\theta} \delta\theta} + \cancel{i} \lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{\cancel{i} r e^{i\theta} \delta\theta}$$

$$= \frac{1}{re^{i\theta}} \left\{ \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{i\,\delta\theta} + \lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{\delta\theta} \right\}$$

[NOTE: i is in the denominator, ∴ we need to rationalize]

$$\boxed{\frac{1}{i} = -i}$$

$$f'(z) = \frac{1}{re^{i\theta}} \left\{ -i \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{\delta\theta} + \lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{\delta\theta} \right\}$$

$$= \frac{1}{re^{i\theta}} \left[-i \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} \right]$$

$$= \frac{e^{-i\theta}}{r} \left[-i \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} \right]$$

$$f'(z) = \frac{e^{-i\theta}}{r} \left[\frac{\partial v}{\partial \theta} - i \frac{\partial u}{\partial \theta} \right] \text{———} ③$$

equate the RHS of ② & ③

$$\cancel{e^{-i\theta}}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]=\frac{\cancel{e^{-i\theta}}}{r}\left[\frac{\partial v}{\partial\theta}-i\frac{\partial u}{\partial\theta}\right]$$

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=\frac{1}{r}\left[\frac{\partial v}{\partial\theta}-i\frac{\partial u}{\partial\theta}\right]$$

equating the real & imaginary parts

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta} \quad \text{and} \quad \frac{\partial v}{\partial r}=\frac{-1}{r}\frac{\partial u}{\partial\theta}$$

(or) $u_r=\frac{1}{r}v_\theta$ and $v_r=\frac{-1}{r}u_\theta$

$$\boxed{r u_r=v_\theta} \quad \text{and} \quad \boxed{r v_r=-u_\theta}$$

They are known as Cauchy Riemann equation in polar form.

Properties of Analytic functions

Harmonic function :-

A function ϕ is said to be harmonic if it satisfies Laplace's equation $\nabla^2\phi=0$

In the Cartesian form $\phi(x,y)$ is harmonic if $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$

In the polar form $\phi(r,\theta)$ is harmonic if $\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$

No Need

Harmonic property:

statement: The real and imaginary parts of an analytic function are harmonic.

Proof:

In Cartesian form

Let $f(z) = u(x,y) + iv(x,y)$ be analytic we shall show that u and v satisfy Laplace equation in the Cartesian form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Since $f(z)$ is analytic we have Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—} \quad ①$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{—} \quad ②$$

Differentiating ① w.r.to x & ② w.r.to y partially we get

$$① \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x \partial y}$$

$$② \Rightarrow \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

but $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ is always true &

hence we have

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{(or)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic}$$

Again differentiating ① w.r. to y and ② w.r. to x partially we get

① ⇒ $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ and

② ⇒ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$

but $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is always true & hence we have

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \therefore \ v \text{ is harmonic}$$

Thus we have proved real & imaginary parts of analytic function when expressed in the Cartesian form satisfies the laplace's equation in Cartesian form.

No Need

In Polar form:

Let $f(z) = u(r,\theta) + i v(r,\theta)$ be analytic. we shall show that u & v satisfies Laplace equations in polar form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

we have Cauchy-Riemann equations in polar form

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{—①}$$

$$r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \quad —— ②$$

D. ① w.r. to r and ② w.r. to θ parti

① ⇒ $$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial^2 v}{\partial r\,\partial \theta}$$

② ⇒ $$r\frac{\partial^2 v}{\partial \theta\,\partial r} = -\frac{\partial^2 u}{\partial \theta^2} \Rightarrow$$

$$\frac{\partial^2 v}{\partial \theta\,\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

But $\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r}$ is always true we have

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

÷ by r on B.S

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = -\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∴ u satisfies the laplace equation in polar form

∴ u is <u>harmonic</u>

D. ① partially w.r. to θ and ② w.r. to r we have

① ⇒ $r\frac{\partial^2 u}{\partial \theta\,\partial r} = \frac{\partial^2 v}{\partial \theta^2} \Rightarrow \frac{\partial^2 u}{\partial \theta\,\partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}$

② ⇒ $r\cdot\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} = -\frac{\partial^2 u}{\partial r\,\partial \theta} \Rightarrow -r\frac{\partial^2 v}{\partial r^2} - \frac{\partial v}{\partial r} = \frac{\partial^2 u}{\partial r\,\partial \theta}$

but $\frac{\partial^2 u}{\partial \theta\,\partial r} = \frac{\partial^2 u}{\partial r\,\partial \theta}$ is always true &

we have

$$\frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} = -r\frac{\partial^2 v}{\partial r^2} - \frac{\partial v}{\partial r}$$

÷ B.S by r

$$\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = -\frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0$$

V satisfies Laplace equation in polar fo

∴ V is harmonic

Thus we have proved u and v are harmonic

Problems

TYPE-1: Finding the derivative of an analytic function.

Working procedure

① Given $\omega = f(z)$, we substituting $z = x+iy$ or $z = re^{i\theta}$ to find the real and imaginary parts u and v as a function x, y (or) r, θ.

② we find first order partial derivatives and verify C-R equations in the Cartesian or polar form to conclude that f(z) is analytic

③ To find the derivative of f(z) we make use of the fundamental results derived while establishing C-R equations They are as follows

$$f'(z) = u_x + iv_x \quad [\text{Cartesian form}]$$

$$f'(z) = e^{-i\theta}(u_r + iv_r) \quad [\text{polar form}]$$

④ we substitute for the partial derivatives and rearrange as a function of $x+iy$ (or) $re^{i\theta}$ which is z, with the result $f'(z)$ is obtained as a function of z.

① Show that $\omega = z + e^z$ is analytic and hence find $\dfrac{d\omega}{dz}$

Soln: By data $\omega = z + e^z$

$$u + iv = x + iy + e^{x+iy}$$

$$= x + iy + e^x \cdot e^{iy}$$

$$u + iv = x + iy + e^x(\cos y + i\sin y)$$

$$u + iv = x + iy + e^x\cos y + ie^x\sin y$$

$$= (x + e^x\cos y) + i(y + e^x\sin y)$$

$u = x + e^x \cos y \qquad V = y + e^x \sin y$

$u_x = 1 + e^x \cos y \qquad V_x = e^x \sin y$

$u_y = -e^x \sin y \qquad V_y = 1 + e^x \cos y$

W.K.T C-R eqn in the Cartesian form $u_x = V_y$ and $V_x = -u_y$ are satisfied

Thus $w = z + e^z$ is analytic

we have $\frac{dw}{dz} = f'(z) = u_x + iV_x$

$$\frac{dw}{dz} = (1 + e^x \cos y) + i(e^x \sin y)$$

$$= 1 + e^x (\cos y + i \sin y)$$

$$= 1 + e^x e^{iy}$$

$$= 1 + e^{x+iy}$$

$$\frac{dw}{dz} = 1 + e^z \qquad \text{Since } z = x + iy$$

② S.T $f(z) = \cosh z$ is analytic and hence find $f'(z)$

Sol^{n}: $f(z) = \cosh z$ $\qquad \cosh z = \cos iz$

$u + iv = \cosh(x + iy) \qquad i^2 = -1$

$= \cos i(x + iy)$ $\qquad$ { Since $\cosh x = \cos ix$, $\sin ix = i \sinh x$

$= \cos(ix + i^2 y)$

$= \cos(ix - y)$

W.K.T $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= \cos ix \cos y + \sin ix \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y \qquad V = \sinh x \sin y$

$u_x = \sinh x \cos y$ $\quad v_x = \cosh x \sin y$
$u_y = -\cosh x \sin y$ $\quad v_y = \sinh x \cos y$

$\therefore$ C-R eqns $u_x = v_y$ & $v_x = -u_y$ are satisfied

Thus f(z) is analytic

we have $\frac{dw}{dz} = f'(z) = u_x + iv_x$

$f'(z) = \sinh x \cos y + i \cosh x \sin y$

$\times^{ly}$ & ÷ by i in the RHS we have

$f'(z) = \frac{1}{i}\left[i \sinh x \cos y + i^2 \cosh x \sin y\right]$

$= \frac{1}{i}\left[i \sinh x \cos y - \cosh x \sin y\right]$

$= \frac{1}{i}\left[\sin ix \cos y - \cos ix \sin y\right]$

It is in the form of

$\sin A \cos B - \cos A \sin B = \sin(A-B)$

where $A = ix$ and $B = y$

$= \frac{1}{i}\left[\sin(ix - y)\right]$

$= \frac{1}{i} \sin i(x+iy)$

W.K.T $\sin ix = i \sinh x$

$= \frac{1}{\cancel{i}} \times \cancel{i} \sinh(x+iy)$

$= \sinh(x+iy)$ $\quad$ since $z = x+iy$

$f'(z) = \sinh z$

$\therefore \frac{dw}{dz} = \sinh x$

③ S.T $f(z) = z^n$, where n is a +ve integer is analytic and hence find its derivative

Sol^n: $f(z) = z^n$

Take $z = re^{i\theta}$ we have

$u + iv = (re^{i\theta})^n$

$u + iv = r^n (e^{i\theta})^n$

$= r^n e^{in\theta}$ $\left\{ \begin{array}{l} \text{W.K.T} \\ e^{i\theta} = \cos\theta + i\sin\theta \end{array} \right.$

$u + iv = r^n (\cos n\theta + i\sin n\theta)$

$u + iv = r^n \cos n\theta + i r^n \sin n\theta$

$u = r^n \cos n\theta$ $\quad v = r^n \sin n\theta$

$u_r = nr^{n-1}\cos n\theta$ $\quad v_r = nr^{n-1}\sin n\theta$

$u_\theta = -r^n n \sin n\theta$ $\quad v_\theta = nr^n \cos n\theta$

C-R eqn in polar form

$r u_r = v_\theta$ & $r v_r = -u_\theta$

are satisfied

	$u_r = nr^{n-1}\cos n\theta$
$v_r = nr^{n-1}\sin n\theta$	
$rv_r = r(nr^{n-1}\sin n\theta)$	$ru_r = r(nr^{n-1}\cos n\theta)$
$= nr^{\cancel{1+n-1}}\sin n\theta$	$ru_r = nr^{\cancel{1+n-1}}\cos n\theta$
$= \underline{nr^n \sin n\theta}$	$ru_r = nr^n \cos n\theta$
$rv_r = -u_\theta$	$ru_r = v_\theta$

Thus $f(z) = z^n$ is analytic

we have $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$f'(z) = e^{-i\theta}\left(nr^{n-1}\cos n\theta + i\,nr^{n-1}\sin n\theta\right)$$

$$= nr^{n-1}e^{-i\theta}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1}e^{-i\theta}\,e^{in\theta}$$

$$= nr^{n-1}\,e^{i\theta(-1+n)}$$

$$= nr^{n-1}\,e^{i\theta(n-1)}$$

$$= n\underline{r^{n-1}}\,\underline{(e^{i\theta})}^{n-1}$$

$$= n(re^{i\theta})^{n-1}$$

$f'(z) = nz^{n-1}$ Since $z = re^{i\theta}$

$$\therefore f'(z) = nz^{n-1}$$

④ S.T $f(z) = \sin z$ is analytic and hence find $f'(z)$.

H.W

Soln: By data

$f(z) = \sin z$

$f(z) = \sin(x+iy)$

W.K.T $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$f(z) = \sin x\cos iy + \cos x\sin iy$

W.K.T $\sin i\theta = i\sinh\theta$

$\cos i\theta = \cosh\theta$

$u+iv = \sin x\cosh y + \cos x\, i\sinh y$

$u+iv = \sin x\cosh y + i\cos x\sinh y$

$u = \sin x \cosh y$ & $V = \cos x \sinh y$

$u_x = \cos x \cosh y$ & $V_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $V_y = \cos x \cosh y$

C-R equations $u_x = V_y$ and $V_x = -u_y$ are satisfied

Thus $f(z)$ is analytic

we have $f'(z) = u_x + iV_x$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$= \cos x \cosh y - i \sin x \sinh y$

$= \cos x \cosh y - \sin x (i \sinh y)$

$f'(z) = \cos x \cos iy - \sin x \sin(iy)$

$\cos A \cos B - \sin A \sin B = \cos(A+B)$

$= \cos(x+iy)$

$f'(z) = \cos z$ Since $z = x+iy$

⑤ S.T $w = \log z$, $z \neq 0$ is analytic & hence find $\frac{dw}{dz}$

Solⁿ By data

$w = \log z$

Taking $z = re^{i\theta}$

$u + iv = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta \log_e e$ $(\log_e e = 1)$

$u + iv = \log r + i\theta$

$u = \log r$ $V = \theta$

$u_r = \frac{1}{r}$ $V_r = 0$

$u_\theta = 0$ $V_\theta = 1$

C-R equations in the polar form
$r u_r = V_\theta$ & $r V_r = -u_\theta$ are satisfied
Thus $f(z)$ ⓞⓡ $w = \log z$ is analytic
we have

$$f'(z) = e^{-i\theta}(u_r + iV_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i(0)\right)$$

$$= e^{-i\theta}\left(\frac{1}{r}\right)$$

$$= \frac{1}{r e^{i\theta}}$$

$$f'(z) = \frac{1}{z}$$

⑥ S.T $w = z\bar{z}$ is not analytic

Soln. By data $w = z\bar{z}$
$z = x + iy$
$\bar{z} = x - iy$

$$u + iv = (x+iy)(x-iy)$$
$$= x^2 - i^2 y^2$$
$$= x^2 - (-1)y^2$$
$$= x^2 + y^2$$

$u = x^2 + y^2 \quad v = 0$

$u_x = 2x, \ u_y = 2y \ ; \quad v_x = 0, \ v_y = 0$

C-R eqns $u_x = v_y$ and $v_x = -u_y$ are not satisfied.

$w = z\bar{z}$ is not analytic

⑦ S.T $f(z) = \left(r + \frac{K^2}{r}\right)\cos\theta + i\left(r - \frac{K^2}{r}\right)\sin\theta$, $r \neq 0$ is a regular function of $z = re^{i\theta}$ also find $f'(z)$

Soln: $f(z) = \left(r + \frac{K^2}{r}\right)\cos\theta + i\left(r - \frac{K^2}{r}\right)\sin\theta$

$u + iv = \left(r + \frac{K^2}{r}\right)\cos\theta + i\left(r - \frac{K^2}{r}\right)\sin\theta$

$u = \left(r + \frac{K^2}{r}\right)\cos\theta \quad v = \left(r - \frac{K^2}{r}\right)\sin\theta$

$u_r = \left(1 + K^2\left(\frac{-1}{r^2}\right)\right)\cos\theta \quad v_r = \left(1 - K^2\left(\frac{-1}{r^2}\right)\right)\sin\theta$

$u_r = \left(1 - \frac{K^2}{r^2}\right)\cos\theta \quad v_r = \left(1 + \frac{K^2}{r^2}\right)\sin\theta$

$u_\theta = \left(r + \frac{K^2}{r}\right)(-\sin\theta) \quad v_\theta = \left(r - \frac{K^2}{r}\right)\cos\theta$

$ru_r = \left(r - \frac{K^2}{r}\right)\cos\theta \quad rv_r = \left(r + \frac{K^2}{r}\right)\sin\theta$

C-R eqns $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied

Thus $f(z)$ is analytic

we have $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= e^{-i\theta}\left[\left(1-\frac{K^2}{r^2}\right)\cos\theta + i\left(1+\frac{K^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[\cos\theta - \frac{K^2}{r^2}\cos\theta + i\sin\theta + i\frac{K^2}{r^2}\sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{K^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{K^2}{r^2}e^{-i\theta}\right]$$

$$= e^{-i\theta}.e^{i\theta} - \frac{K^2}{r^2}e^{-i\theta}.e^{-i\theta}$$

$$= e^{-i\cancel{\theta}+i\cancel{\theta}} - \frac{K^2}{r^2}\left(e^{-2i\theta}\right)$$

$$= e^{0} - \frac{K^2}{r^2}e^{-2i\theta}$$

$$= 1 - \frac{K^2}{r^2e^{2i\theta}}$$

$$= 1 - \frac{K^2}{r^2(e^{i\theta})^2}$$

$$= 1 - \frac{K^2}{(re^{i\theta})^2}$$

$f'(z) = 1 - \frac{K^2}{z^2}$ since $z = re^{i\theta}$

TYPE - ②

Construction of analytic function $f(z)$ given its real or imaginary part

working. procedure

① Given u or v as a function of x, y we find u_x, u_y or v_x, v_y and Consider $f'(z) = u_x + i v_x$

② Given u, we use C-R equation $v_x = -u_y$ or given v we use $u_x = v_y$ So that $f'(z) = u_x - i u_y$ (or) $f'(z) = v_y + i v_x$

③ we substitute the expression for the partial derivatives in the RHS and then put $x = z, y = 0$ to obtain $f'(z)$ as a function of z.

④ Integrating w.r. to z we get $f(z)$

⑤ III'ly in the Case of polar co-ordinates r, θ we consider

$f'(z) = e^{-i\theta} (u_r + i v_r)$ & use C-R equation in the RHS, $v_r = -\frac{1}{r} u_\theta$ given u

or $u_r = \frac{1}{r} v_\theta$ given v.

⑥ we use the Substitution $r = z, \theta = 0$ to obtain $f'(z)$ as a function of z

⑦ Integrate w.r. to z we get $f(z)$.

This method is known as milne's thomson method

① Construct the analytic function whose real part is $u = \log\sqrt{x^2+y^2}$

Solⁿ: $u = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2} = \frac{1}{2}\log(x^2+y^2)$

D. w. r. to x

$$u_x = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}x$$

$$u_x = \frac{x}{x^2+y^2}$$

$$u = \frac{1}{2}\log(x^2+y^2)$$

D. w. r. to y

$$u_y = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}y$$

$$u_y = \frac{y}{x^2+y^2}$$

Consider

$f'(z) = u_x + iv_x$ But $v_x = -u_y$ (C-R eqⁿ)

$= u_x - iu_y$

$$f'(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \quad \text{—— ①}$$

put $x = z$ & $y = 0$ we have

$$f'(z) = \frac{z}{z^2+0} - i(0)$$

$$= \frac{\cancel{z}}{\cancel{z^2}}$$

$$f'(z) = 1/z$$

Integrate w.r.to z

$$f(z) = \int \frac{1}{z}\, dz + C$$

$$f(z) = \log z + C$$

② Find the analytic function $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$

Soln: $V = e^x(x\sin y + y\cos y)$

$$V_x = e^x(\sin y + 0) + (x\sin y + y\cos y)e^x$$

$$V_x = e^x(\sin y + x\sin y + y\cos y) \text{ — ①}$$

$$V_y = e^x(x\cos y + y(-\sin y) + \cos y) + (x\sin y + y\cos y)(0)$$

$$V_y = e^x(x\cos y - y\sin y + \cos y) \text{ — ②}$$

Consider $f'(z) = u_x + iV_x$ But $u_x = V_y$ (C-R eqn)

$$f'(z) = V_y + iV_x$$

$$f'(z) = e^x(x\cos y - y\sin y + \cos y) + i(e^x(\sin y + x\sin y + y\cos y))$$

put $x = z$, $y = 0$ we have

$$f'(z) = e^z(z\cos(0) - 0 + \cos(0)) + i(e^z(0 + 0 + 0))$$

$$= e^z(z(1) + 1) + i(0) \quad \left\{ \begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array} \right.$$

$$f'(z) = e^z(z+1)$$

Integrate w.r.to z

$$f(z) = \int e^z(z+1)\, dz + C$$

$$= \int (z+1)e^z\, dz + C$$

$$= (z+1)e^z - \int e^z \cdot \frac{d}{dz}(z+1)\, dz + C$$

$$= (z+1)e^z - e^z(1) + C$$

$$= ze^z + \cancel{e^z} - \cancel{e^z} + C$$

$$f(z) = ze^z + C$$

③ Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solⁿ:

$u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}(\cos 2y - 0) + (x\cos 2y - y\sin 2y)(2e^{2x})$

$u_x = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$

$u_y = e^{2x}(-2x\sin 2y - \{2y\cos 2y + \sin 2y\}) + (x\cos 2y - y\sin 2y)(0)$

$u_y = -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y) + 0$

$u_y = -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y)$

consider $f'(z) = u_x + iv_x$ $(v_x = -u_y)$

$f'(z) = u_x - iu_y$ by CR eqⁿ

$f'(z) = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y) - i(-e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y))$

put $x = z,\ y = 0$

$f'(z) = e^{2z}(\cos(0) + 2z\cos(0) - 0) + i(e^{2z}(2z\cancel{\sin(0)}^{\,0} + 2\cancel{(0)\cos(0)}^{\,0} + \cancel{\sin(0)}^{\,0}))$

$= e^{2z}(1 + 2z) + i(0)$

$f'(z) = e^{2z}(1 + 2z)$

Integrate w.r. to z

$f(z) = \int (1+2z)e^{2z}\,dz + C$

$= (1+2z)\dfrac{e^{2z}}{2} - \int \dfrac{e^{2z}}{2} \cdot \dfrac{d}{dz}(1+2z)\,dz + C$

$= (1+2z)\dfrac{e^{2z}}{2} - \dfrac{1}{2}\left[\dfrac{e^{2z}}{\cancel{2}} \times \cancel{2}\right] + C$

$f(z) = \dfrac{e^{2z}}{2} + \cancel{2}z\dfrac{e^{2z}}{\cancel{2}} - \dfrac{1}{2}e^{2z} + C$

$f(z) = \cancel{\dfrac{e^{2z}}{2}} + ze^{2z} - \cancel{\dfrac{e^{2z}}{2}} + C$

$\underline{\underline{f(z) = ze^{2z}}} + C$

Thus $f(z) = ze^{2z} + C$

Also $f(z) = u + iV$

~~$= e^{2z}\ (z)$~~

$$= (x+iy)\,e^{2(x+iy)}$$

$$= (x+iy)\,e^{2x+i2y}$$

$$= (x+iy)\,e^{2x} \cdot e^{i2y}$$

$$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$$

$$= e^{2x}(x\cos 2y + iy\cos 2y + ix\sin 2y + i^2 y\sin 2y)$$

$$= e^{2x}(x\cos 2y + i(y\cos 2y + x\sin 2y) - y\sin 2y)$$

$$= e^{2x}(\underline{x\cos 2y - y\sin 2y} + i(y\cos 2y + x\sin 2y))$$

$$f(z) = e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(y\cos 2y + x\sin 2y)$$

④ Find the analytic function whose real part is $\frac{x^4 - y^4 - 2x}{x^2+y^2}$, hence find V

Sol^n:

$$u = \frac{x^4 - y^4 - 2x}{x^2+y^2}$$

D. w. r. to x

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

where $u' = du/dx$

$v' = \frac{dv}{dx}$

$$u_x = \frac{(x^2+y^2)(4x^3 - 0 - 2) - (x^4 - y^4 - 2x)(2x)}{(x^2+y^2)^2}$$

$$u_x = \frac{(x^2+y^2)(4x^3-2) - (x^4-y^4-2x)(2x)}{(x^2+y^2)^2}$$

D.U. w.r. to y

$$u_y = \frac{(x^2+y^2)(-4y^3) - (x^4-y^4-2x)(2y)}{(x^2+y^2)^2}$$

Consider $f'(z) = u_x + i V_x$

but $V_x = -u_y$ (CR eqⁿ)

$$f'(z) = u_x - i u_y$$

$$f'(z) = \left\{\frac{(x^2+y^2)(4x^3-2) - (x^4-y^4-2x)2x}{(x^2+y^2)^2}\right\} - i\left\{\frac{(x^2+y^2)(-4y^3) - (x^4-y^4-2x)(2y)}{(x^2+y^2)^2}\right\}$$

put $x = z,\ y = 0$

$$f'(z) = \left\{\frac{(z^2+0)(4z^3-2) - (z^4-0-2z)2z}{(z^2)^2}\right\} - i\left\{\frac{(z^2+0)(0) - (z^4-0-2z)(0)}{(z^2)^2}\right\}$$

$$f'(z) = \left\{\frac{z^2(4z^3-2) - (2z^5-4z^2)}{z^4}\right\} - i\left\{\frac{0-0}{z^4}\right\}$$

$$f'(z) = \frac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4} - i(0)$$

$$= \frac{2z^5 + 2z^2}{z^4}$$

$$= \frac{2z^5}{z^4} + \frac{2z^2}{z^4}$$

$$f'(z) = 2z + \frac{2}{z^2} = 2\left(z + \frac{1}{z^2}\right)$$

$$f(z) = 2\int\left(z + \frac{1}{z^2}\right) dz + C$$

$$= 2\left[\frac{z^2}{2} - \frac{1}{z}\right] + C$$

$$f(z) = z^2 - \frac{2}{z} + C$$

Now we $_\wedge$have to find V

$$f(z) = u + iv = (x+iy)^2 - \frac{2}{x+iy}$$

$$= (x^2 + i^2y^2 + 2xiy) - \frac{2(x-iy)}{(x+iy)(x-iy)}$$

$$= (x^2 - y^2 + i2xy) - \frac{2(x-iy)}{x^2+y^2}$$

$$= (x^2-y^2) + 2xiy - \frac{2(x-iy)}{x^2+y^2}$$

$$(x+iy)(x-iy)$$
$$= x^2 - i^2y^2$$
$$= x^2 - (-1)y^2$$
$$= x^2 + y^2$$

$$= (x^2-y^2) + 2xiy - \frac{2x + 2iy}{x^2+y^2}$$

$$= (x^2-y^2) + 2xiy - \frac{2x}{x^2+y^2} + \frac{i2y}{x^2+y^2}$$

$$= \left(x^2 - y^2 - \frac{2x}{x^2+y^2}\right) + i\left(2xy + \frac{2y}{x^2+y^2}\right)$$

$$= \left(\frac{x^4 + \cancel{x^2y^2} - \cancel{y^2x^2} - y^4 - 2x}{x^2+y^2}\right) + i\left(\frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}\right)$$

$$u+iv = \left(\frac{x^4 - y^4 - 2x}{x^2+y^2}\right) + i\left(\frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}\right)$$

$\therefore$ Imaginary part

v is $\dfrac{2x^3y + 2xy^3 + 2y}{x^2+y^2}$ //

⑤ Construct the analytic function whose real part is $r^2\cos 2\theta$.

Sol^n: $u = r^2\cos 2\theta$

$u_r = 2r\cos 2\theta$, $u_\theta = -2r^2\sin 2\theta$

consider

$f'(z) = e^{-i\theta}(u_r + iv_r)$

~~$f'(z) = e^{-i\theta}(u(2r\cos 2\theta) + i(-2r^2\sin 2\theta))$~~

but $v_r = -\dfrac{1}{r}u_\theta$ (C-R eq^n)

~~$= e^{-i\theta}(2r\cos 2\theta + i$~~

$f'(z) = e^{-i\theta}\left(u_r + i\left(\dfrac{-1}{r}\right)u_\theta\right)$

$= e^{-i\theta}\left(2r\cos 2\theta + i \times \dfrac{-1}{r} \times -2r^2\sin 2\theta\right)$

$f'(z) = e^{-i\theta}(2r\cos 2\theta + i\,2r\sin 2\theta)$

$f'(z) = 2re^{-i\theta}(\cos 2\theta + i\sin 2\theta)$

put $r = z$ & $\theta = 0$ we have

$f'(z) = 2ze^{-i(0)}(\cos(0) + i\sin(0))$

$= 2ze^{0}\ (1+0)$

$f'(z) = 2z(1)(1)$

$f'(z) = 2z$

Integrate w.r. to z

$$f(z) = 2\int z \cdot dz + C$$

$$= \cancel{2}\cdot\frac{z^2}{\cancel{2}} + C$$

$$\underline{\underline{f(z) = z^2 + C}}$$

How ⑥ Find the analytic function $f(z)$ whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Ans: $f(z) = \cot z + C$

⑦ S.T the following function u is harmonic. Also determine the corresponding analytic function $f(z)$.

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

Sol: $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$ — ①

$u_y = +\sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$ — ②

① + ② $\Rightarrow u_{xx} + u_{yy} = -\cancel{\sin x \cosh y} - \cancel{2\cos x \sinh y} + \cancel{2} + \cancel{\sin x \cosh y} + \cancel{2\cos x \sinh y} - \cancel{2}$

$$\underline{\underline{u_{xx} + u_{yy} = 0}}$$

Thus u is harmonic

Consider $f'(z) = u_x + iv_x$ But $v_x = -u_y$ (CR eqⁿ)

$f'(z) = u_x - iu_y$

$$f'(z) = [\cos x \cosh y - 2\sin x \sinh y + 2x + 4y] - i[\sin x \sinh y + 2\cos x \cosh y - 2y + 4x]$$

put $x = z$ $y = 0$

$$= [\cos z \cos(0) - 2\sin z \sin(0) + 2z + 4(0)] - i[\sin z \sin(0) + 2\cos(z)\cos(0) - 2(0) + 4z]$$

$$f'(z) = \cos z + 2z - i(2\cos z + 4z)$$

Integrate w.r. to z

$$f(z) = \int \{(\cos z + 2z) - i(2\cos z + 4z)\} dz + C$$

$$f(z) = \sin z + \frac{\cancel{2}z^2}{\cancel{2}} - i\left(2\sin z + \frac{4z^2}{2}\right) + C$$

$$= \sin z + z^2 - i(2\sin z + 2z^2) + C$$

$$f(z) = \sin z + z^2 - i2\sin z - i2z^2 + C$$

$$f(z) = \sin z(1 - 2i) + z^2(1 - 2i) + C$$

$$f(z) = \underline{\underline{(1 - 2i)(}} z^2 + \sin z) + C$$

OR

Q Determine the analytic function $f(z)$ whose imaginary part is $\left(r-\frac{k^2}{r}\right)\sin\theta$, $r \neq 0$, hence find the real part of $f(z)$ & P.T it is harmonic

Sol:

$$V = \left(r - \frac{k^2}{r}\right)\sin\theta$$

$$V_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$$

$$V_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Consider $f'(z) = e^{-i\theta}(u_r + iV_r)$

but $u_r = \frac{1}{r} V_\theta$ (C-R eqn)

$$f'(z) = e^{-i\theta}\left(\frac{1}{r}V_\theta + iV_r\right)$$

$$= e^{-i\theta}\left(\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$$

$$= e^{-i\theta}\left(\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$$

put $r = z$ & $\theta = 0$

$$= e^{0}\left\{\left(1 - \frac{k^2}{z^2}\right)\cos 0 + i\,\cancel{\left(1 + \frac{k^2}{z^2}\right)\sin(0)}\right\}$$

$\to 0$

$$= (1)\left(1 - \frac{k^2}{z^2}\right)(1)$$

$$f'(z) = 1 - \frac{k^2}{z^2}$$

Int. w.r. to z

$$f(z) = \int \left(1 - \frac{K^2}{z^2}\right) dz + C$$

$$f(z) = z - K^2\left(-\frac{1}{z}\right) + C$$

$$\underline{f(z) = z + \frac{K^2}{z} + C}$$

Now let us find real part u
take $z = re^{i\theta}$

$$f(z) = u + iv = re^{i\theta} + \frac{K^2}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{K^2}{r} e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{K^2}{r}(\cos\theta - i\sin\theta)$$

$$= r\cos\theta + ir\sin\theta + \frac{K^2}{r}\cos\theta - i\frac{K^2}{r}\sin\theta$$

$$u = \cos\theta\left(r + \frac{K^2}{r}\right) + i\sin\theta\left(r - \frac{K^2}{r}\right)$$

$$\therefore \text{ real part } u = \underline{\underline{\cos\theta\left(r + \frac{K^2}{r}\right)}}$$

Now we have to S.T u is harmonic

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{—①}$$

$$u = \cos\theta\left(r + \frac{K^2}{r}\right)$$

$$u_r = \cos\theta\left(1 - \frac{K^2}{r^2}\right)$$

$$u_{rr} = \cos\theta\left(+\frac{2K^2}{r^3}\right)$$

$$u_\theta = -\sin\theta\left(r + K^2/r\right)$$

$$u_{\theta\theta} = -\cos\theta\left(r + K^2/r\right)$$

Now consider LHS of ①

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$= \cos\theta\left(\frac{2k^2}{r^3}\right) + \frac{1}{r}\left(\cos\theta\left(1-\frac{k^2}{r^2}\right)\right) + \frac{1}{r^2}\left(-\cos\theta\left(r+k^2/r\right)\right)$$

$$= \frac{2\cos\theta \cdot k^2}{r^3} + \cos\theta\left(\frac{1}{r}-\frac{k^2}{r^3}\right) - \cos\theta\left(\frac{1}{r}+k^2/r^3\right)$$

$$= \underline{\frac{2\cos\theta \cdot k^2}{r^3}} \cancel{+\frac{1}{r}\cos\theta} + \underline{\cos\theta \cdot \frac{k^2}{r^3}} \cancel{-\frac{1}{r}\cos\theta} - \cos\theta \cdot \frac{k^2}{r^3}$$

$$= \cancel{\frac{\cos\theta \cdot k^2}{r^3}} \cancel{-\frac{k^2}{r^3}\cos\theta}$$

$$= 0$$

TYPE-3 Finding the Conjugate harmonic function & analytic function

We have proved that the real and imaginary parts of an analytic function $f(z) = u+iv$ are harmonic. u and v are called Conjugate harmonic functions. Given u we can find v & Via Versa

Working procedure

① Given u we can find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

② We consider C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

③ Substituting for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ we obtain a system of two non homogeneous PDE of the form $\frac{\partial v}{\partial y} = f(x,y)$ &

$$\frac{\partial v}{\partial x} = g(x,y)$$

④ These can be solved by direct integration to obtain required V.

⑤ The same proceduce is adopted to find u given V.

⑥ Further u+iV will give us f(z) as a function of x, y. putting x = z, y=0 we obtain f(z) as a function of z

① S.T $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function f(z).

Solⁿ:

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$u_{xx} = 6x + 6$$

$$u_y = -6xy - 6y$$

$$u_{yy} = -6x - 6$$

$$\therefore \quad u_{xx} + u_{yy} = \cancel{6}x + \cancel{6} - 6\cancel{x} - \cancel{6}$$

$$u_{xx} + u_{yy} = 0$$

Thus u is harmonic

Now Consider C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Substituting for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$$

Integrate w.r. to y

$$V = \int (3x^2 - 3y^2 + 6x)\,dy + f(x)$$

$u_x = \partial u/\partial x$
$u_y = \partial u/\partial y$
$v_x = \partial v/\partial x$
$v_y = \frac{\partial v}{\partial y}$

$$V = 3x^2y - \cancel{3}y^3/\cancel{3} + 6xy + f(x)$$

$$\frac{\partial v}{\partial x} = -(-6xy - 6y)$$

$$\frac{\partial v}{\partial x} = 6xy + 6y$$

Integrate w.r. to x

$$V = \int (6xy + 6y)\,dx + g(y)$$

$$= 6y \cdot \frac{x^2}{2} + 6yx + g(y)$$

$$V = 3x^2y + 6yx + g(y)$$

Now we have to properly choose $f(x)$ and $g(y)$ to obtain a unique expression for V.

$$\therefore f(x) = 0, \; g(y) = -y^3$$

$$V = 3x^2y - y^3 + 6xy + 0$$

$$\underline{V = 3x^2y - y^3 + 6xy}$$

The analytic function is $f(z) = u+iv$

$$f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy)$$

put $x = z,\ y = 0$

$$f(z) = (z^3 - 0 + 3z^2 - 0 + 1) + i(0 - 0 + 0)$$

$$f(z) = z^3 + 3z^2 + 1 \,//$$

② S.T $V = \log\sqrt{x+y}$ is harmonic

Soln:

$$V = \log\sqrt{x+y}$$

$$V = \log(x+y)^{1/2}$$

$$V = \tfrac{1}{2}\log(x+y)$$

$$V_x = \frac{1}{2} \times \frac{1}{x+y}, \qquad V_y = \frac{1}{2} \times \frac{1}{x+y}$$

$$V_{xx} = -\frac{1}{2} \times \frac{1}{(x+y)^2} \qquad V_{yy} = -\frac{1}{2} \times \frac{1}{(x+y)^2}$$

$$V_{xx} + V_{yy} = \frac{-1}{2(x+y)^2} - \frac{1}{2(x+y)^2}$$

$$= \frac{-\cancel{2}}{\cancel{2}(x+y)^2}$$

$$= \frac{-1}{(x+y)^2} \neq 0$$

$\therefore$ V is not harmonic

③ S.T $V = \cos x \sinh y$ is harmonic. Find the Conjugate harmonic function and express $u+iV$ as a analytic function of Z.

Sol^{n}: $V = \cos x \sinh y$

$V_x = -\sin x \sinh y$

$V_{xx} = -\cos x \sinh y$

$V_y = \cos x \cosh y$

$V_{yy} = \cos x \, \cancel{\cos}\sinh y$

$V_{xx} + V_{yy} = -\cos x \sinh y + \cos x \sinh y$

$V_{xx} + V_{yy} = 0$

To find the harmonic conjugate, we Consider C-R eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Substituting for $\underset{V_y \leftarrow}{\frac{\partial v}{\partial y}}$ and $\frac{\partial v}{\partial x} \rightarrow V_x$,

$\frac{\partial u}{\partial x} = \cos x \cosh y$

Integrate w.r. to x

$u = \int \cos x \cosh y \, . dx + f(y)$

$u = \cosh y \sin x + f(y)$

$\frac{\partial u}{\partial y} = -(-\sin x \sinh y)$

$\partial u/\partial y = \sin x \sinh y$

$u = \int (\sin x \sinh y) dy + g(x)$

$u = \sin x \cosh y + g(x)$

To get a unique expression for u choose $f(x) = 0$ & $g(x) = 0$

$u = \cosh y \sin x$

$f(z) = u + iv$

$= \sin x \cosh y + i \cos x \sinh y$

put $x = z$, $y = 0$ we get

$f(z) = \sin z \cos(0) + i \cancelto{0}{\cos(0)\sin(0)}$

$f(z) = \sin z\,(1)$

$\underline{f(z) = \sin z}$

④ Given that $u = x^2 + 4x - y^2 + 2y$ as the real part of an analytic function find V and hence find f(z) in terms of z.

↳ we have to find harmonic conjugate

They are not asking to S.T u is harmonic

Soln:

$u = x^2 + 4x - y^2 + 2y$

u_x or $\dfrac{\partial u}{\partial x} = 2x + 4$

u_y or $\dfrac{\partial u}{\partial y} = -2y + 2$

Consider C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\dfrac{\partial v}{\partial y} = 2x + 4$

$v = \int (2x+4)\,dy + f(x)$

$V = 2xy + 4y + f(x)$

$\dfrac{\partial v}{\partial x} = -(-2y + 2)$

$\dfrac{\partial v}{\partial x} = 2y - 2$

$v = \int (2y - 2)\,dx + g(y)$

$v = 2yx - 2x + g(y)$

To get unique expression for v

choose $f(x) = -2x,\ g(y) = 4y$

$$\therefore v = 2xy + 4y - 2x$$

$$f(z) = u + iv$$

$$= (x^2+4x-y^2+2y) + i(2xy+4y-2x)$$

put $x = z,\ y = 0$

$$= (z^2+4z-0+0) + i(0+0-2z)$$

$$f(z) = \underline{(z^2+4z) - 2iz}$$

⑤ S.T $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic. Find its harmonic conjugate and also the corresponding analytic function.

Solⁿ: $u = \left(r + \frac{1}{r}\right)\cos\theta$

we shall S.T $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ — ①

$u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta$; $u_{rr} = \left(0 + \frac{2}{r^3}\right)\cos\theta$

$$u_{rr} = +\frac{2}{r^3}\cos\theta$$

$u_\theta = \left(r + \frac{1}{r}\right)(-\sin\theta)$; $u_{\theta\theta} = \left(r+\frac{1}{r}\right)(-\cos\theta)$

Consider the LHS of ①

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$= +\frac{2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{1}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(\cos\theta\left(r + \frac{1}{r}\right)\right)$$

$$= +\frac{2}{r^3}\cos\theta + \left(\frac{1}{r} - \frac{1}{r^3}\right)\cos\theta - \cos\theta\left(\frac{1}{r} + \frac{1}{r^3}\right)$$

$= \cancel{\frac{+2}{r^3}\cos\theta} + \cancel{\frac{\cos\theta}{r}} - \cancel{\frac{\cos\theta}{r^3}} - \cancel{\frac{\cos\theta}{r}} - \cancel{\frac{\cos\theta}{r^3}}$

$= 0$ ∴ u is harmonic

To find V, let us consider C-R eqns in polar form

$r u_r = v_\theta \quad ; \quad r v_r = -u_\theta$

$v_\theta = r\left(1 - \frac{1}{r^2}\right)\cos\theta$

$\frac{\partial v}{\partial \theta} \leftarrow v_\theta = \left(r - \frac{1}{r}\right)\cos\theta$

$V = \int \left(r - \frac{1}{r}\right)\cos\theta \, d\theta + f(r)$

$V = \left(r - \frac{1}{r}\right)(\sin\theta) + f(r)$

$v_r = -\frac{1}{r} u_\theta$

$= -\frac{1}{r}\left\{\left(r + \frac{1}{r}\right)(-\sin\theta)\right\}$

$v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta$

$\hookrightarrow \frac{\partial v}{\partial r}$

$V = \int \left(1 + \frac{1}{r^2}\right)\sin\theta \, dr + g(\theta)$

$V = \sin\theta\left(r - \frac{1}{r}\right) + g(\theta)$

To get unique expression for V let us choose $f(r) = 0$ & $g(\theta) = 0$

$V = \left(r - \frac{1}{r}\right)\sin\theta$

we have $f(z) = u + iv$

$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

put $r = 2, \theta = 0$

$f(z) = \left(z + \frac{1}{2}\right)\cos(0) + i\,\cancel{\left(r - \frac{1}{r}\right)\sin(0)} \rightarrow 0$

$f(z) = z + \frac{1}{2}$ is analytic function

H.W ⑥ S.T $u = e^{x}(x\cos y - y\sin y)$ is harmonic and find its harmonic Conjugate. Also find corresponding analytic function.

Solⁿ:

$$u = e^{x}(x\cos y - y\sin y)$$

$$u_x = e^{x}(\cos y - 0) + (x\cos y - y\sin y)e^{x}$$

$$u_x = e^{x}(\cos y + x\cos y - y\sin y)$$

$$u_{xx} = e^{x}(0 + \cos y - 0) + (\cos y + x\cos y - y\sin y)e^{x}$$

$$u_{xx} = e^{x}(\cos y + \cos y + x\cos y - y\sin y)$$

$$u_{xx} = e^{x}(2\cos y + x\cos y - y\sin y)$$

$$u_y = e^{x}(-x\sin y - y\cos y - \sin y) + \underbrace{(x\cos y - y\sin y)(0)}_{0}$$

$$u_y = -e^{x}(+x\sin y + y\cos y + \sin y)$$

$$u_{yy} = -e^{x}(x\cos y - y\sin y + \cos y + \cos y) + \underbrace{(x\sin y + y\cos y + \sin y)(0)}_{0}$$

$$u_{yy} = -e^{x}(x\cos y - y\sin y + 2\cos y)$$

$$u_{xx} + u_{yy} = \cancel{e^{x}(2\cos y + x\cos y - y\sin y)}$$
$$\cancel{- e^{x}(2\cos y + x\cos y - y\sin y)}$$

$$u_{xx} + u_{yy} = 0$$

$\therefore$ u is harmonic

$$\int I\,II\,dx = I\int II\,dx - \int\int II\,\frac{d}{dx}(I)\,dx$$

Now Consider C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = e^{x}(\cos y + x\cos y - y\sin y)$$

$$V = \int e^{x}(\cos y + x\cos y - y\sin y)\,dy + f(x)$$

$y)e^{x}$

$$V = e^{x}\left[\sin y + x\cdot\sin y - \left\{y(-\cos y) - \int -\cos y\,(1)\,dy\right\}\right] + f(x)$$

Integration by parts

$$V = e^{x}\left[\cancel{\sin y} + x\sin y + y\cos y - \cancel{\sin y}\right] + f(x)$$

$y\sin y)(0)$

$$V = xe^{x}\sin y + e^{x}y\cos y + f(x)$$

$$\frac{\partial v}{\partial x} = -\left(-e^{x}(x\sin y + y\cos y + \sin y)\right)$$

$$\frac{\partial v}{\partial x} = e^{x}(x\sin y + y\cos y + \sin y)$$

$$V = \int e^{x}(x\sin y + y\cos y + \sin y)\,dx + g(y)$$

$$= \sin y\int xe^{x}dx + y\cos y\int e^{x}dx + \sin y\int e^{x}dx + g(y)$$

$$= \sin y\left[xe^{x} - \int e^{x}(1)\,dx\right] + y\cos y\,(e^{x}) + \sin y\,(e^{x}) + g(y)$$

$$= \sin y\,(xe^{x} - e^{x}) + e^{x}y\cos y + e^{x}\sin y + g(y)$$

$$= xe^{x}\sin y - \cancel{e^{x}\sin y} + e^{x}y\cos y + \cancel{e^{x}\sin y} + g(y)$$

$v = xe^x \sin y + e^x y \cos y + g(y)$

To get unique expression for V
choose $f(x) = 0$, $g(y) = 0$

$V = xe^x \sin y + e^x y \cos y$

$v = e^x(x \sin y + y \cos y)$

$f(z) = u + iv$

$f(z) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y)$

put $x = z$ $y = 0$

$= e^z(z\cos(0) - 0) + ie^z(0 + 0)$

$= e^z(z) + 0$

$f(z) = ze^z$ is analytic function

TYPE - II :

① Find the analytic function $f(z) = u + iv$
given $u - v = e^x(\cos y - \sin y)$

Solⁿ: $u - v = e^x(\cos y - \sin y)$

D. partially w.r.to x

$u_x - v_x = e^x(\cancelto{0}{0-0}) + (\cos y - \sin y)e^x$

$u_x - v_x = e^x(\cos y - \sin y)$ — ①

lly $u_y - v_y = e^x(-\sin y - \cos y) + 0$

$u_y - v_y = -e^x(\sin y + \cos y)$ — ②

Using C-R equations for the LHS of eqn ②

$u_y = -v_x$ and $v_y = u_x$ we have

$$-v_x - u_x = -e^x(\sin y + \cos y)$$

$$+(v_x + u_x) = +e^x(\sin y + \cos y)$$

$$v_x + u_x = e^x(\sin y + \cos y) \quad \text{— ③}$$

Solve ① & ③

$$u_x - \cancel{v_x} = e^x(\cos y - \sin y)$$

$$\cancel{v_x} + u_x = e^x(\sin y + \cos y)$$

$$2u_x = e^x(\cos y - \cancel{\sin y} + \cancel{\sin y} + \cos y)$$

$$\cancel{2}u_x = \cancel{2}e^x \cos y$$

$$u_x = e^x \cos y$$

① - ③

$$u_x - v_x = e^x(\cos y - \sin y)$$

$$-(v_x + u_x) = -e^x(\sin y + \cos y)$$

$$\Rightarrow \cancel{u_x} - v_x = e^x(\cos y - \sin y)$$

$$-v_x - \cancel{u_x} = -e^x(\sin y + \cos y)$$

$$-2v_x = e^x(\cancel{\cos y} - \sin y - \sin y - \cancel{\cos y})$$

$$-\cancel{2}v_x = -\cancel{2}e^x \sin y$$

$$v_x = e^x \sin y$$

$$f'(z) = u_x + i v_x$$

$$f'(z) = e^x \cos y + i e^x \sin y$$

put $x = z,\ y = 0$

$$= e^z \cos(0) + i\cancel{e^z \sin(0)} \to 0$$

$$f'(z) = e^z(1) \Rightarrow \underline{f'(z) = e^z}$$

Integrate w.r.to z

$$\underline{f(z) = e^z + C}$$

(29) If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic and given that $u+v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

$r \neq 0$ determine the analytic function $f(z)$.

Solⁿ: $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

D. partially w.r.to r & θ we have

$$u_r + v_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta) \quad \text{—①}$$

$$u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta) \text{—②}$$

Using C-R equations:

$$v_\theta = r u_r \quad \text{and} \quad -u_\theta = r v_r$$

② ⇒ $-rV_r + rU_r = -\frac{2}{r^2}(\sin 2\theta + \cos 2\theta)$ — ③

Solve ① and ③

$$U_r + V_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$r(-V_r + U_r) = -\frac{2}{r^2}(\sin 2\theta + \cos 2\theta)$$

$$\Rightarrow U_r + \cancel{V_r} = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$\cancel{-V_r} + U_r = -\frac{2}{r^3}(\sin 2\theta + \cos 2\theta)$$

$$\cancel{2}U_r = -\frac{\cancel{2}}{r^3}(\cos 2\theta - \cancel{\sin 2\theta} + \cancel{\sin 2\theta} + \cos 2\theta)$$

$$U_r = -\frac{1}{r^3} \times 2\cos 2\theta$$

① - ③

$$U_r + V_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$r(V_r - U_r) = \frac{2}{r^2}(\sin 2\theta + \cos 2\theta)$$

$$\Rightarrow \cancel{U_r} + V_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$$

$$V_r - \cancel{U_r} = \frac{2}{r^3}(\sin 2\theta + \cos 2\theta)$$

$$\cancel{2}V_r = \frac{\cancel{2}}{r^3}(-\cancel{\cos 2\theta} + \sin 2\theta + \sin 2\theta + \cancel{\cos 2\theta})$$

$$V_r = \frac{1}{r^3} \times 2\sin 2\theta$$

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\frac{-1}{r^3} \times 2\cos 2\theta + i\frac{1}{r^3} \times 2\sin 2\theta\right)$$

$$= \frac{-2e^{-i\theta}}{r^3}(\cos 2\theta - i\sin 2\theta)$$

$$= -\frac{2e^{-i\theta}}{r^3}(e^{-i2\theta})$$

$$= -\frac{2}{r^3}e^{-i3\theta}$$

$$= \frac{-2}{r^3 e^{i3\theta}} = \frac{-2}{r^3(e^{i\theta})^3} = \frac{-2}{(re^{i\theta})^3}$$

$$f'(z) = \frac{-2}{z^3}$$

$$f(z) = -2\int \frac{1}{z^3}\,dz + C$$

$$f(z) = -\cancel{2}\left(\frac{-1}{\cancel{2}z^2}\right) + C$$

$$\underline{f(z) = \frac{1}{z^2} + C}$$

#③ If $f(z)$ is analytic S.T,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

Soln: Let $f(z) = u + iv$ be analytic

$\therefore \; |f(z)| = \sqrt{u^2+v^2}$

(or) $|f(z)|^2 = u^2+v^2 = \phi$ (say)

To prove that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

i.e to prove that $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

consider

$\phi = u^2+v^2$ and D. w.r. to x partially

$\phi_x = 2uu_x + 2vv_x$

$\phi_x = 2[uu_x + vv_x]$

D. w.r. to x again

$\phi_{xx} = 2[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x]$

$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$ — ①

111ly we can get

$\phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$ — ②

adding ① & ②

$$\phi_{xx}+\phi_{yy} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2 + uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$$

$$= 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \quad — ③$$

Since $f(z) = u+iv$ is analytic, u and v are harmonic

$u_{xx}+u_{yy} = 0, \quad v_{xx}+v_{yy} = 0$

$$= 2[u0 + v0 + u_x^2 + v_x^2 + \underbrace{(-v_x)^2 + (u_x)^2}_{\text{CR eqn}}]$$

$$= 2\left[u_x^2 + v_x^2 + v_x^2 + u_x^2\right]$$

$$= 2\left[2u_x^2 + 2v_x^2\right]$$

$$\phi_{xx}+\phi_{yy} = 4\left[u_x^2 + v_x^2\right] \quad — ④$$

$$f'(z) = u_x + i v_x$$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

or

$$|f'(z)|^2 = u_x^2 + v_x^2$$

using this in the RHS of ④ we have

$$\phi_{xx}+\phi_{yy} = 4|f'(z)|^2$$

$$\phi_{xx} + \phi_{yy} = 4\,|f'(z)|^2 \,//$$

Q④ If $f(z)$ is a regular function of z S.T $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

Sol^n: Let $f(z) = u+iv$ be the regular (analytic) function

$$|f(z)| = \sqrt{u^2+v^2} = \phi \text{ (say)}$$

$$|f(z)|^2 = u^2 + v^2$$

we have to P.T $\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = |f'(z)|^2$

$$\phi_x^2 + \phi_y^2 = |f'(z)|^2$$

where $\phi = \sqrt{u^2+v^2}$

Consider $\phi^2 = u^2+v^2$

D. w. r. to x partially

$$2\phi\phi_x = 2uu_x + 2vv_x$$

$$\cancel{2}\phi\phi_x = \cancel{2}(uu_x+vv_x)$$

$$\phi\phi_x = uu_x + vv_x \quad — ①$$

III^ly we get

$$\phi\phi_y = uu_y + vv_y \quad — ②$$

Squaring & adding ① & ② we have

$$(\phi\phi_x)^2+(\phi\phi_y)^2 = (uu_x+vv_x)^2+(uu_y+vv_y)^2$$

$$\phi^2\phi_x^2+\phi^2\phi_y^2 = (u^2u_x^2+v^2v_x^2+2uu_xvv_x)+(u^2u_y^2+v^2v_y^2+2uu_yvv_y)$$

$$\phi^2(\phi_x^2+\phi_y^2) = (u^2u_x^2+v^2v_x^2+2uvu_xv_x)+(u^2v_x^2+v^2u_x^2+2uv(-v_x)(u_x))$$

Since $f(z)=u+iv$ is analytic we used C-R eq^ns

$u_y=-v_x$ & $v_y=u_x$

in the RHS of second bracket

$$\phi^2(\phi_x^2+\phi_y^2) = u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)$$

$$\phi^2(\phi_x^2+\phi_y^2) = (u_x^2+v_x^2)(u^2+v^2)$$

$\phi^2=u^2+v^2$ & using this in the RHS we have

$$\phi^2(\phi_x^2+\phi_y^2) = \phi^2(u_x^2+v_x^2)$$

$$\phi_x^2 + \phi_y^2 = u_x^2 + v_x^2 \quad \text{——} ③$$

But $f'(z) = u_x + i v_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

using this in the RHS of ③ we have

$$\underline{\underline{\phi_x^2 + \phi_y^2 = |f'(z)|^2}}$$

⑤ If $\phi(x,y)$ is a differentiable function and $f(z) = u(x,y) + iv(x,y)$ is a regular function, show that

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \left\{\left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\phi}{\partial v}\right)^2\right\} |f'(z)|^2$$

Solⁿ: ϕ is a composite function by regarding ϕ to be a function of u and v where u and v are functions of x, y.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x} \; ; \; \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$$

$\phi_x = \phi_u u_x + \phi_v \cdot v_x$; $\phi_y = \phi_u u_y + \phi_v v_y$

S on B.S ; S on B.S

$\phi_x^2 = (\phi_u u_x + \phi_v v_x)^2$; $(\phi_y)^2 = (\phi_u u_y + \phi_v v_y)^2$

$$\phi_x^2 = \phi_u^2 u_x^2 + \phi_v^2 v_x^2 + 2\phi_u u_x \phi_v \phi_x$$

use C-R eqⁿ in $(\phi_y)^2$

C-R eqⁿ
$u_y = -v_x$
$v_y = u_x$

$$(\phi_y)^2 = (-\phi_u v_x + \phi_v u_x)^2$$

$$\phi_y^2 = \phi_u^2 V_x^2 + \phi_v^2 u_x^2 - 2\phi_u V_x \phi_v u_x$$

$$\phi_x^2 + \phi_y^2 = \phi_u^2 u_x^2 + \phi_v^2 V_x^2 + \cancel{2\phi_u u_x \phi_v V_x}$$

$$+ \phi_u^2 V_x^2 + \phi_v^2 u_x^2 \cancel{- 2\phi_u V_x \phi_v u_x}$$

$$\phi_x^2 + \phi_y^2 = \phi_u^2 (u_x^2 + V_x^2) + \phi_v^2 (u_x^2 + V_x^2)$$

$$\phi_x^2 + \phi_y^2 = (u_x^2 + V_x^2)(\phi_u^2 + \phi_v^2) \quad \text{—} ①$$

but $f'(z) = u_x + iV_x$

$$|f'(z)|^2 = u_x^2 + V_x^2$$

now ① becomes

$$\phi_x^2 + \phi_y^2 = (\phi_u^2 + \phi_v^2)\,|f'(z)|^2$$

module-02

Conformal transformations and Complex integration

Bilinear Transformation (BLT)

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are real constants ∋; ad - bc ≠ 0 is called a bilinear transformation.

working Procedure :

⇒① Given w_1, w_2, w_3 corresponding to z_1, z_2, z_3 we assume the billinear transformation in the form $w = \frac{az+b}{cz+d}$

② we substitute the given set of points to obtain a set of three equations in four unknowns a, b, c, d.

③ we deduce a pair of equation in any three unknowns and solve by the rule of cross x^n to obtain a proportionate set of values for three unknowns.

④ The values are used to find the fourth unknown.

⑤ All these four values when substituted in the assumed form of ω, will give us the required bilinear transformation.

Problems

① Find the bilinear transformation which map the points $z = 1, i, -1$ into $\omega = i, 0, -i$.

Solⁿ: Let $\omega = \frac{az+b}{cz+d}$ — ① be required bilinear transformation

when $z = 1$, $\omega = i$

eqⁿ ① becomes

$$i = \frac{a+b}{c+d}$$

$$a+b = ci+di$$

$$a+b-ci-di = 0 \quad — ②$$

when $z = i,\ \omega = 0$

eqn ① becomes

$$0 = \frac{ai+b}{ci+d}$$

$$ai+b = 0 \quad — ③$$

when $z = -1,\ \omega = -i$

eqn ① becomes

$$-i = \frac{-a+b}{-c+d}$$

$$ci - di = -a+b$$

$$-a+b-ci+di = 0 \quad — ④$$

Solve ② & ④

$$\cancel{a}+b-ci-\cancel{di} = 0$$

$$-\cancel{a}+b-ci+\cancel{di} = 0$$

$$2b-2ci = 0$$

$$2(b-ic) = 0$$

$$b-ic = 0 \quad — ⑤$$

by cross x^n method $\quad ai+b+0(c) = 0$

solve ③ & ⑤ $\quad a(0)+b-ic = 0$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{a}{-i} = \frac{-b}{1} = \frac{c}{i} = K \text{ (say)}$$

$a = -iK \quad -b = K \quad c = iK$

$b = -K$

put a, b, c in ②

$$-iK - K - i^2K - di = 0$$

$$-iK - \cancel{K} + \cancel{K} - di = 0$$

$$-d\cancel{i} = \cancel{i}K$$

$$d = -K$$

put a, b, c, d in ω

$$\omega = \frac{-iKz - K}{iKz - K} = \frac{\cancel{-K}(+iz+1)}{\cancel{-K}(-iz+1)}$$

$$\omega = \frac{1+iz}{1-iz}$$

② Find the B.L.T which map the points $z = 1, i, -1$ into $\omega = 2, i, -2$ also find the invariant points of the transformation.

Sol^n: $\omega = \dfrac{az+b}{cz+d}$ — ①

when $z=1,\ \omega=2$

Now ① becomes

$$2=\frac{a+b}{c+d}$$

$$a+b=2c+2d$$

$$a+b-2c-2d=0 \quad -②$$

when $z=i,\ \omega=i$

① becomes

$$i=\frac{ai+b}{ci+d}$$

$$ci^2+di=ai+b$$

$$-c+di=ai+b$$

$$ai+b+c-di=0 \quad -③$$

when $z=-1,\ \omega=-2$

① becomes

$$-2=\frac{-a+b}{-c+d}$$

$$2c-2d=-a+b$$

$$-a+b-2c+2d=0 \quad -④$$

~~Solve ② & ③~~

~~$a+b-2c-2d=0\times i$~~

~~$ai+b+c-di=0$~~

~~$ai+bi-2ci-2di=0$~~

~~$(-)\,ai \underset{(-)}{+} b \underset{(-)}{+} c \underset{(+)}{-} di=0$~~

~~$b(i-1)-c(2i$~~

Solve ② & ④

$$\cancel{a}+b-2c-2\cancel{d}=0$$
$$-\cancel{a}+b-2c+\cancel{2}d=0$$
$$2b-4c=0$$
$$2(b-2c)=0$$
$$b-2c=0 \quad — ⑤$$

Solve ③ & ④

$$ai+b+c-di=0$$
$$-a+b-2c+2d=0 \times i$$
$$\cancel{a}i+b+c-di=0$$
$$-\cancel{a}i+bi-2ci+2di=0$$
$$b(1+i)+c(1-2i)+id=0 \quad - ⑥$$

Solve ⑤ & ⑥ by cross x^n method

$$b-2c+0d=0$$
$$b(1+i)+c(1-2i)+id=0$$

$$\frac{b}{\begin{vmatrix}-2 & 0\\ 1-2i & i\end{vmatrix}}=\frac{-c}{\begin{vmatrix}1 & 0\\ 1+i & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}1 & -2\\ 1+i & 1-2i\end{vmatrix}}$$

$$\frac{b}{-2i-0}=\frac{-c}{i-0}=\frac{d}{1-\cancel{2i}+2+\cancel{2i}}=K$$

$$\frac{b}{-2i}=\frac{-c}{i}=\frac{d}{3}=K$$

$b = -2iK, \quad -c = iK, \quad d = 3K$

$c = -iK$

put b, c, d in ②

$a - \cancel{2i}K + \cancel{2iK} - 6K = 0$

$a - 6K = 0$

$a = 6K$ //

put all values in ω

$$\omega = \frac{6KZ - 2iK}{-iKZ + 3K}$$

$$= \frac{\cancel{K}(6z - 2i)}{\cancel{K}(-iz + 3)}$$

$$\omega = \frac{6z - 2i}{-iz + 3} //$$

To find the invariant points of this transformation are obtained by taking $\omega = Z$.

$$z = \frac{6z - 2i}{-iz + 3}$$

$-iz^2 + 3z = 6z - 2i$

$-iz^2 + 3z - 6z + 2i = 0$

$-iz^2 - 3z + 2i = 0$

$iz^2 + 3z - 2i = 0$

$$Z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{9-4 \times i \times -2i}}{2i}$$

$$= \frac{-3 \pm \sqrt{9+8i^2}}{2i}$$

$$= \frac{-3 \pm \sqrt{9-8}}{2i} \qquad (i^2=-1)$$

$$= \frac{-3 \pm \sqrt{1}}{2i}$$

$$= \frac{-3 \pm 1}{2i}$$

$$Z = \frac{-3+1}{2i}, \qquad Z = \frac{-3-1}{2i}$$

$$Z = \frac{-\cancel{2}}{\cancel{2}i} \qquad Z = \frac{-4}{2i}$$

$$Z = \frac{-1}{i} \qquad Z = \frac{-2}{i}$$

$$Z = -(-i) \qquad Z = -2\left(\frac{1}{i}\right)$$

$$Z = i \,// \qquad Z = -2(-i)$$

$$Z = 2i \,//$$

$$\boxed{\begin{aligned} \frac{1}{i} &= \frac{1}{i} \times \frac{-i}{-i} \\ &= \frac{-i}{-i^2} \\ &= \frac{-i}{-(-1)} = -i \\ \therefore \frac{1}{i} &= -i \end{aligned}}$$

∴ Invariant points are

$i, 2i$

③ find the B.L.T which maps $z_1=-1$, $z_2=0$, $z_3=1$ into $w_1=0$, $w_2=i$, $w_3=3i$

Soln:

$$w=\frac{az+b}{cz+d} \quad \text{— ①}$$

when $w=0$ $z=-1$

① becomes

$$0=\frac{-a+b}{-c+d}$$

$$-a+b=0 \quad \text{— ②}$$

when $w=i$, $z=0$

$$i=\frac{0+b}{0+d} \Rightarrow b=id \Rightarrow b-id=0 \text{—③}$$

when $w=3i$, $z_3=1$

① becomes

$$3i=\frac{a+b}{c+d}$$

$$3ci+3di=a+b$$

$$a+b-3ci-3di=0 \quad \text{— ④}$$

Solve ② & ③

$$-a+\cancel{b}=0$$

$$\underset{(-)}{}\ \cancel{b}\underset{(+)}{-}di=0$$

$$-a+di=0 \quad \text{— ⑤}$$

Solve ③ & ⑤ by cross x^n method

$0(a) + b - id = 0$

$-a + 0(b) + id = 0$

$$\frac{a}{\begin{vmatrix} 1 & -i \\ 0 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -i \\ -1 & i \end{vmatrix}} = \frac{c}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}}$$

$$\frac{a}{i} = \frac{-b}{-i} = \frac{d}{1} = K$$

$a = i \quad -b = -iK \quad d = K$

$a = i, \quad b = iK \quad d = K$

put in ③

$iK + iK - 3ci - 3Ki = 0$

$-3ci - Ki = 0$

$-i(3c + K) = 0$

$3c + K = 0$

$3c = -K$

$c = -K/3$

put all values in ω

$$w = \frac{izK + iK}{-\frac{K}{3}z + K}$$

$$w = \frac{i\cancel{K}(z+1)}{\cancel{K}(-z/3+1)}$$

$$w = \frac{i(z+1)}{(-z/3+1)} \,//$$

(or)

$$w = \frac{i(z+1)}{\left(\frac{-z+3}{3}\right)}$$

$$w = \frac{3i(z+1)}{-z+3} \,//$$

④ Find the B.L.T which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of transformations. (Invariant points)

Sol^n:

$$w = \frac{az+b}{cz+d}$$

$$w = \frac{z(a+b/z)}{z(c+d/z)}$$

$$w = \frac{a+b/z}{c+d/z} \quad — \circledast$$

when $z = \infty$, $w = -1$

⊛ becomes $-1 = \frac{a+0}{c+0}$

$$-1 = \frac{a}{c} \Rightarrow a = -c \Rightarrow a+c = 0 \quad ①$$

when $z = i,\ w = -i$

⊛ becomes $-i = \frac{a + b/i}{c + d/i}$

$$-i = \frac{\frac{ai+b}{i}}{\frac{ci+d}{i}}$$

$$-i = \frac{ai+b}{ci+d}$$

$$-i(ci) - di = ai + b$$

$$-ci^2 - di = ai + b$$

$$-c(-1) - di = ai + b$$

$$c - di = ai + b$$

$$ai + b - c + di = 0 \quad ②$$

when $z = 0,\ w = 1$

⊛ becomes $1 = \frac{b}{d}$

$$b = d \Rightarrow b - d = 0 \quad — ③$$

Solve ① & ②

$$a + 0(b) + \cancel{c} + 0d = 0$$

$$\underline{ai + b - \cancel{c} + di = 0}$$

$$a(1+i) + b + di = 0 \quad — ④$$

by Cross x^n solve ③ & ④

$$a(0) + b - d = 0$$

$$a(1+i) + b + di = 0$$

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{0+1+i} = \frac{d}{-(1+i)} = K$$

$a = K(1+i)$ $\quad -b = K(+1+i)$, $d = -K(1+i)$

$a = K(1+i)$ $\quad b = -K(1+i)$

$b = -K(1+i)$

put a, b, d valuey in ②

$$K(1+i)i - K(1+i) - c - K(1+i)i = 0$$

$$K(i+i^2) + K(1+i) - c + K(i+i^2) = 0$$

$i^2 = -1$

$$k(i-1) - k(1+i) - c - k(i-1) = 0$$

$$\cancel{ik} - \cancel{k} - k - \cancel{ik} - c - ik + \cancel{k} = 0$$

$$-k - c - ik = 0$$

$$-c - ik - k = 0$$

$$-c = k + ik$$

$$c = -k - ik$$

$$\underline{\underline{c = -k(1+i)}}$$

put a, b, c, d in w

$$w = \frac{az+b}{cz+d}$$

$$= \frac{k(1+i)z + (-k)(1+i)}{-k(1+i)z - k(1+i)}$$

$$= \frac{\cancel{k(1+i)}(z-1)}{\cancel{-k(1+i)}(-z-1)}$$

$$= \frac{\cancel{-}(1-z)}{\cancel{-}(z+1)}$$

$$w = \frac{1-z}{1+z} \;//$$

To find invariant points take $w = z$ in ⊛

$$z = \frac{1-z}{1+z}$$

$$z + z^2 = 1 - z$$

$$z^2 + z + z - 1 = 0$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$z = -1 \pm \sqrt{2}$$

∴ invariant points are $-1+\sqrt{2}$, $-1-\sqrt{2}$ //

⑤ Do yourself

Find the B.L.T which maps the points $z = 1, i, -1$ into $w = 0, 1, \infty$

$$w = \frac{az+b}{cz+d}$$

$z=1, \; w=0;$

(*) becomes

$$0 = \frac{a+b}{c+d}$$

$$a+b=0 \quad — ①$$

$z=i, \; w=1; \quad 1 = \dfrac{ai+b}{ci+d}$

$$ai+b-ci-d=0 \quad — ②$$

$z=-1, \; w=\infty$

consider $\dfrac{1}{w} = \dfrac{cz+d}{az+b}$, $\left(\dfrac{1}{\infty}=0\right)$ $\left(\because \dfrac{1}{w} = \dfrac{1}{\infty} = 0\right)$

$$0 = \frac{-c+d}{-a+b}$$

$$-c+d=0 \quad — ③$$

solve ② & ③

$$\begin{aligned} ai+b-ci-\cancel{d} &= 0 \\ -c+\cancel{d} &= 0 \\ \hline ai+b-c(i+1) &= 0 \quad — ④ \end{aligned}$$

Solving ① & ④ by Cross x^n method

$$a+b+0c=0$$

$$ai+b-c(i+1)=0$$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -(i+1) \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 0 \\ a & -(i+1) \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ i & 1 \end{vmatrix}}$$

$$\frac{a}{-(i+1)} = \frac{-b}{-(i+1)} = \frac{c}{1-i} = K$$

$$\frac{a}{-(i+1)} = K, \quad \frac{b}{i+1} = K, \quad \frac{c}{1-i} = K$$

$a = -K(i+1) \quad b = K(i+1) \quad c = K(1-i)$

put a, b, c in ②

① $-K(i+1)i + K(i+1) - K(1-i)i - d = 0$

$$-Ki^2 - Ki + \cancel{Ki} + K + Ki^2 - \cancel{Ki} - d = 0$$

$$-K(-1) - iK + K + K(-1) - d = 0$$

$$+\cancel{K} - iK + K - \cancel{K} - d = 0$$

$$K(1-i) - d = 0$$

$$\underline{d = K(1-i)}$$

put all values in ω

$$\omega = \frac{-\cancel{K}\{(1+i)z + (1+i)\}}{\cancel{K}\{(1-i)z + (1-i)\}}$$

$$\omega = \frac{(1+i)(1-z)}{(1-i)(1+z)}$$

$\times^{ly}$ & ÷ by $1+i$

$$\omega = \frac{(1+i)(1-z)(1+i)}{(1-i)(1+z)(1+i)}$$

$$= \frac{(1+i)^2(1-z)}{(1-i^2)(1+z)}$$

$$= \frac{(1+i^2+2i)(1-z)}{(1+1)(1+z)}$$

$$= \frac{(\cancel{1}-\cancel{1}+2i)(1-z)}{(2)(1+z)}$$

$$w = i\left[\frac{1-z}{1+z}\right] //$$

⑥ Find the B.L.T which map the points $z=0, 1, \infty$ in to the points $w=-5, -1, 3$ respectively. what are the invariant points?

Sol^n: $w = \frac{az+b}{cz+d}$ — (*)

$z=0,\ w=-5$

(*) becomes $-5 = \frac{b}{d}$

$b = -5d$ — (**)

$b+5d=0$ — ①

$z=1,\ w=-1$

(*) becomes $-1 = \frac{a+b}{c+d}$

$-c-d = a+b$

$a+b+c+d=0$ — ②

$z=\infty,\ w=3$

$$w = \frac{\cancel{z}(a+b/z)}{\cancel{z}(c+d/z)}$$

$$w = \frac{(a+b/\infty)}{(c+d/\infty)}$$

$$3 = \frac{a+0}{c+0}$$

$$3 = \frac{a}{c} \Rightarrow a = 3c \quad - (***)$$

$-3c+a=0$ — ③

Solve ② & ③

$a+b+c+d=0$

$(-)\ a \pm 0b \mp 3c \pm 0d = 0$

$b+4c+d=0$ — ④

Solve ① & ④ by cross x^n method

$b+0c+5d=0$

$b+4c+d=0$

$$\frac{b}{\begin{vmatrix}0 & 5\\ 4 & 1\end{vmatrix}} = \frac{-c}{\begin{vmatrix}1 & 5\\ 1 & 1\end{vmatrix}} = \frac{d}{\begin{vmatrix}1 & 0\\ 1 & 4\end{vmatrix}}$$

$$\frac{b}{-20} = \frac{\cancel{-}c}{\cancel{-}4} = \frac{d}{4} = K$$

$\frac{b}{-20} = K, \quad \frac{c}{4} = K, \quad \frac{d}{4} = K$

$b = -20K, \quad c = 4K, \quad d = 4K$

put b, c, d in ②

$a - 20K + 4K + 4K = 0$

$a - 12K = 0$

$a = 12K$ //

$$w = \frac{\cancel{K}(\overset{4\cdot 3}{12z} - 20)}{\cancel{K}(4z + 4\cancel{K})}$$

$$= \frac{\cancel{4}(3z - 5)}{\cancel{4}(z + 1)}$$

$$w = \frac{3z - 5}{z + 1}$$ //

ⓞⓡ use (**) & (***) in ②

② $\Rightarrow$ $3c - 5d + c + d = 0$

$4c - 4d = 0$

$4(c - d) = 0$

$c - d = 0$

$\underline{\underline{c = d}}$

put $c = d$ in ②

$a + b + d + d = 0$

$a + b + 2d = 0$ — ⑤

Solve ① & ⑤ by cross x^n method

$$0(a)+b+5d=0$$

$$a+b+2d=0$$

$$\frac{a}{\begin{vmatrix}1&5\\1&2\end{vmatrix}}=\frac{-b}{\begin{vmatrix}0&5\\1&2\end{vmatrix}}=\frac{d}{\begin{vmatrix}0&1\\1&1\end{vmatrix}}$$

$$\frac{a}{2-5}=\frac{-b}{-5}=\frac{d}{-1}=K$$

$$\frac{a}{-3}=K,\quad \frac{b}{5}=K,\quad \frac{d}{-1}=K$$

$$a=-3K,\quad b=5K,\quad d=-K$$

② $\Rightarrow$ $-3K+5K+c-K=0$

$$K+c=0$$

$$c=-K \,//$$

$$\omega=\frac{\cancel{K}(-3z+5)}{\cancel{K}(-z-1)}$$

$$=\frac{+(3z-5)}{+(z+1)}$$

$$\omega=\frac{3z-5}{z+1}\,//$$

To find invariant points put $w = z$

$$z = \frac{3z-5}{z+1}$$

$$z^2 + z = 3z - 5$$

$$z^2 + z - 3z + 5 = 0$$

$$z^2 - 2z + 5 = 0$$

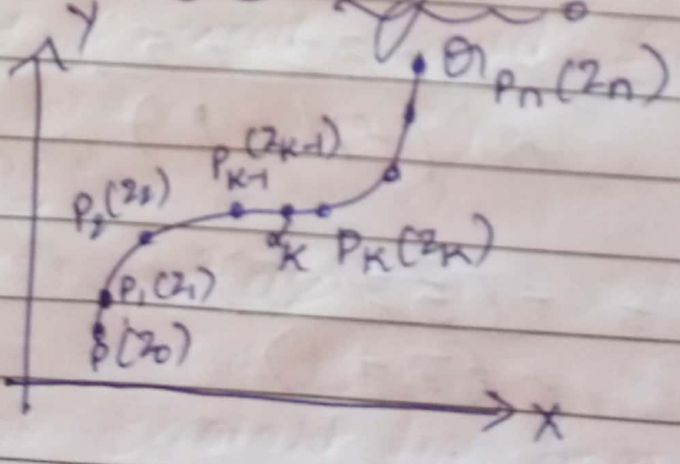
$$z = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm i4}{2}$$

$z = 1 \pm 2i$ are the invariant points.

Complex line integral :



Consider a Contineous function $f(z)$ of the Complex variable $z = x+iy$ defined at all points of a curve

C extending from P to Q. Divide the curve C into n parts by arbitarily taking points $P = P(z_0), P_1(z_1), P_2(z_2) \dots P_K(z_K) \dots P_n(z_n) = Q$ on the curve C. Let α_K be any point on the arc of the curve from P_{K-1} to P_K and let $\delta z_K = z_K - z_{K-1}$ where $K = 1, 2, 3 \dots n$

Then

$$\lim_{n\to\infty} \sum_{K=1}^{n} f(\alpha_K)\,\delta z_K \quad \text{where } \max|\delta z_K| \to 0$$

as $n \to \infty$ is defined as complex line integral along the path C usually denoted by $\int_C f(z)\,dz$.

Properties of Complex integral :

① If $-C$ denotes the curve traversed from Q to P then $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

② If C is split into a no. of parts $C_1, C_2, C_3 \dots$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \dots$$

③ If λ_1 & λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\, dz = \lambda_1 \int_C f_1(z)\, dz \pm \lambda_2 \int_{C_2} f_2(z)\, dz$$

Line integral of a Complex valued function

Let $f(z) = u(x,y) + i v(x,y)$ be a complex valued function defined over a region R and C be a curve in the region Then

$$\int_C f(z)\, dz = \int_C (u+iv)(dx+idy)$$

$$\int_C f(z)\, dz = \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy).$$

This shows that evaluation of a line integral of a complex valued function is nothing but the evaluation of line integrals of real valued functions.

Problems :

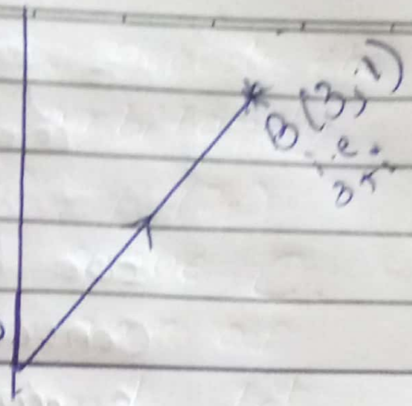
① Evaluate $\int_C z^2\, dz$

ⓐ along the straight line from $z=0$ to $3+i$

ⓑ along the curve madeup of two line segments one from $z=0$ to 3 & another from $z=3$ to $3+i$

) dz

Sol^n: $\int_C z^2 dz = \int_{z=0}^{3+i} z^2 dz$



here Z variy from 0 to 3+i meany (x, y) variy from (0,0) to (3,1).

The eqn of line joining the points (0,0) and (3,1) iy given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-0}{x-0} = \frac{1-0}{3-0}$$

$$\frac{y}{x} = \frac{1}{3} \text{ or } y = \frac{x}{3}$$

$$z^2 = (x+iy)^2$$

$$= x^2 + i^2y^2 + 2xiy$$

$$z^2 = x^2 - y^2 + i2xy$$

$$dz = dx + idy$$

$$\int_C z^2 dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i2xy\}(dx+idy)$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)dx + i(x^2-y^2)dy + i2xydx + i^2 2xy\,dy\}$$

$$= \int_{(0,0)}^{(3,1)} (x^2-y^2)dx + i(x^2-y^2)dy + i2xydx - 2xy\,dy$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)\,dx - 2xy\,dy\} + i\int_{(0,0)}^{(3,1)} \{(x^2-y^2)\,dy + 2xy\,dx\}$$

we have $y = \frac{x}{3}$ or $x = 3y$ & we shall Convert thye integral into the variable y & integrate w.r. to y from 0 to 1 & also $dx = 3dy$

$$\int_C z^2\,dz = \int_{y=0}^{1} \{(9y^2-y^2)\,3dy - 2(3y)y\,dy\} +$$

$$i\int_{y=0}^{1} \{(9y^2-y^2)\,dy + 2(3y)y3dy\}$$

$$= \int_{y=0}^{1} (24y^2\,dy - 6y^2\,dy) + i\int_{y=0}^{1} 8y^2\,dy + 18y^2\,dy$$

$$= \int_{y=0}^{1} 18y^2\,dy + i\int_{y=0}^{1} 26y^2\,dy$$

$$= 18\left[\frac{y^3}{3}\right]_{y=0}^{1} + i26\left[\frac{y^3}{3}\right]_{y=0}^{1}$$

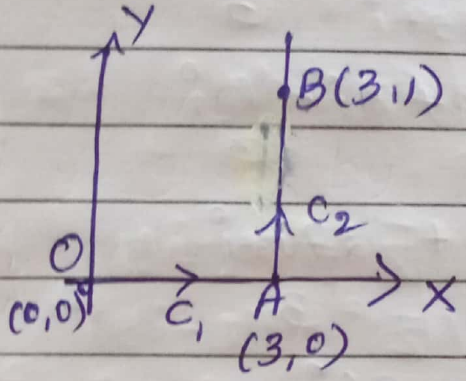
$$= 18\left[\tfrac{1}{3} - 0\right] + i26\left[\tfrac{1}{3} - 0\right]$$

$$= \overset{6}{\cancel{18}}/\cancel{3} + i\,26/3$$

$$= \underline{6 + i\,26/3}$$ along the given path

$2xy\,dx\}$

ⓑ Segments from $z=0$ to $z=3$ and then from $z=3$ to $3+i$ means (x,y) vary from $(0,0)$ to $(3,0)$ & then from $(3,0)$ to $(3,1)$



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \quad \text{—— ①}$$

along C_1 : $y=0 \Rightarrow dy=0$ & x vary from 0 to 3

$z^2 dz$ becomes $x^2 dx$

along C_2 : $x=3 \Rightarrow dx=0$ & y vary from 0 to 1

$z^2 dz$ becomes $(3+iy)^2 i\,dy$.

Now ① becomes

$$\int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} \underbrace{(3+iy)^2}_{(a+b)^2} dy$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{3} + i\int_{y=0}^{1} (9-y^2+6iy)\,dy$$

$(x+iy)^2$

$(3+iy)^2$

$$= 9 + i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1$$

$$= 9 + i\left(9 - \frac{1}{3} + 3i - 0\right)$$

$$= 9 + 9i - \frac{1}{3}i - 3$$

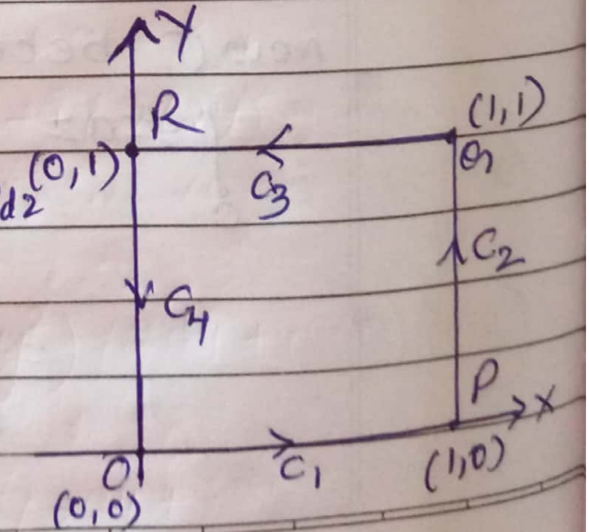
$$= 6 + 9i - \frac{1}{3}i$$

$$= 6 + \frac{27i - i}{3}$$

$$= 6 + \frac{26i}{3}$$

$$\int_C z^2 dz = 6 + i\frac{26}{3} \,//$$ along the given path

③ Evaluate $\int_C |z|^2 dz$ where C is a Square with following vertices (0,0) (1,0) (1,1) (0,1)



Sol^n:

$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad ①$$

we have $|z|^2 dz = \left(\sqrt{x^2+y^2}\right)^2 (dx+idy)$

$$|z|^2 dz = (x^2+y^2)dx + idy$$

along OP (C_1), $y=0 \Rightarrow dy=0$. $|z|^2 dz = x^2 dx$ when $0 \le x \le 1$

along PQ (C_2), $x=1 \Rightarrow dx=0$. $|z|^2 dz = (1+y^2)idy$, $0 \le y \le 1$

along QR (C_3), $y=1 \Rightarrow dy=0$. $|z|^2 dz = (x^2+1)dx$, $1 \le x \le 0$

along RO (C_4), $x=0 \Rightarrow dx=0$. $|z|^2 dz = y^2(idy)$ where $1 \le y \le 0$

use these results in ①

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2 dx + i\int_{y=0}^{1} (1+y^2)dy + \int_{x=1}^{0} (x^2+1)dx$$

$$+ i\int_{y=1}^{0} y^2 dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \left(\frac{1}{3}-0\right) + i\left(1+\frac{1}{3}-0\right) + \left(0-\left(\frac{1}{3}+1\right)+0\right) + i\left(0-\frac{1}{3}\right)$$

$$= \frac{1}{3} + i\left(\frac{4}{3}\right) - \frac{4}{3} + i\left(-\frac{1}{3}\right)$$

$$= \frac{1}{3} + i\frac{4}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= \frac{1+4i-4-i}{3} = \frac{-3+3i}{3} = -1+i$$

$$\int_C |z|^2 dz = \underline{\underline{-1+i}} \text{ along the given path}$$

③ Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along:

ⓐ the line $x = 2y$

ⓑ the real axis upto 2 & then vertically to $2+i$

Sol^{n}: let $I = \int_0^{2+i} (\bar{z})^2 \, dz$

we have $\bar{z} = x - iy$

$$(\bar{z})^2 = (x - iy)^2$$
$$= x^2 + (iy)^2 - 2xiy$$
$$= x^2 + i^2y^2 - 2xiy$$
$$(\bar{z})^2 = x^2 - y^2 - i2xy$$
$$(\bar{z})^2 = (x^2 - y^2) - i(2xy)$$
$$dz = dx + i\,dy$$

ⓐ along $x = 2y$, $dx = 2dy$

$z = 0$ to $2+i \Rightarrow (x, y)$ vary from $(0,0)$ to $(2,1)$ where $0 \le y \le 1$

$$I = \int_{y=0}^{1} \left(\left((2y)^2 - y^2\right) - i\left(2(2y)\,y\right)\right)(2dy + i\,dy)$$

$$= \int_{y=0}^{1} \left(4y^2 - y^2 - i4y^2\right)(2dy + i\,dy)$$

$$= \int_{y=0}^{1} \left(3y^2 - i4y^2\right)(2dy + i\,dy)$$

$$= \int_{y=0}^{1} (3-4i)\,y^2 \cdot dy\,(2+i)$$

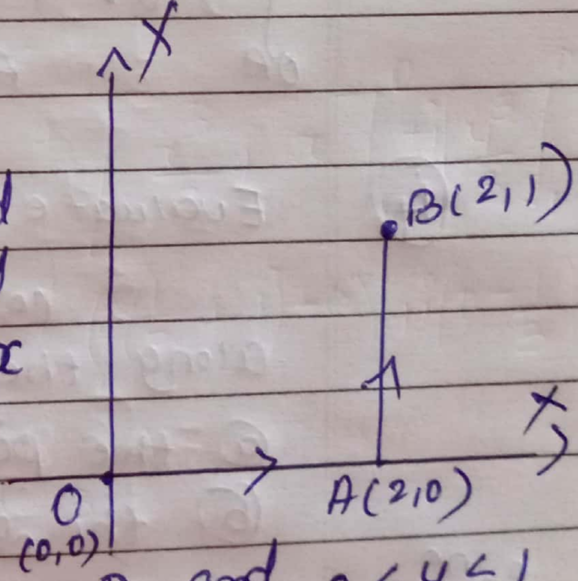
$$= (3-4i)\,(2+i) \int_{y=0}^{1} y^2 \cdot dy$$

$$= (6+3i-8i-4i^2)\left[y^3/3 \right]_0^1$$

$$= (6-5i+4)\left[1/3 - 0 \right]$$

$I = (10-5i)\dfrac{1}{3}$ along the given path.

ⓑ $I = \int_{OA} (\bar{z})^2\, dz + \int_{AB} (\bar{z})^2\, dz$ — ①



Along OA where $O = (0,0)$ and $A = (2,0)$; $y = 0 \Rightarrow dy = 0$ and $0 \le x \le 2$ and $(\bar{z})^2 dz = x^2 dx$

Along AB where $A = (2,0)$ and $B = (2,1)$; $x = 2 \Rightarrow dx = 0$ and $0 \le y \le 1$

$(\bar{z})^2 dz = (2^2 - y^2) - i(4y)$

~~along OA y=0~~

$(\bar{z})^2\, dz = (4-y^2) - i4y$

Now ① becomes

$$\int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{8}{3}$$

$$\int_{AB} (\bar{z})^2 dz = i\int_{y=0}^{1} [(4-y^2) - 4iy]\, dy$$

$$\left[4y - \frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= i\left[\left(4-\frac{1}{3}\right)-0\right] + 4\left[\frac{1}{2} - 0\right]$$

$$\int_{AB} (\bar{z})^2 dz = i\frac{11}{3} + 2$$

∴ ① becomes

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz = 8 + \left(i\frac{11}{3} + 2\right)$$

④ Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$

along the following paths:

ⓐ the parabola $x = 2t$, $y = t^2 + 3$

ⓑ the straight line from (0,3) to (2,4)

Solⁿ:

ⓐ x varies from 0 to 2 and hence

$$\left.\begin{aligned} &\text{If } x=0,\ 2t=0\ \therefore t=0 \\ &\text{If } x=2,\ 2t=2\ \therefore t=1 \end{aligned}\right\} \Rightarrow t \text{ also varies from } 0 \text{ to } 1$$

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$$

$$I = \int_{t=0}^{1} \{2(t^2+3)+4t^2\}\,2dt + \{3(2t)-(t^2+3)\}\,2t\,dt$$

$$= \int_0^1 (2t^2+6+4t^2)\,2dt + (6t-t^2-3)\,2t\,dt$$

$$= \int_0^1 (6t^2+6)\,2dt + (6t-t^2-3)\,2t\,dt$$

$$= \int_0^1 (12t^2+12+12t^2-2t^3-6t)\,dt$$

$$= \int_0^1 (24t^2-2t^3-6t+12)\,dt$$

$$= \left[\frac{\overset{8}{\cancel{24}}\cdot t^3}{\cancel{3}} - \frac{2t^4}{4} - 6\frac{t^2}{2} + 12t\right]_0^1$$

$$= 8[1-0] - \frac{1}{2}(1-0) - 3(1-0) + 12(1-0)$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= 17 - \frac{1}{2}$$

$$= \frac{34-1}{2}$$

$\underline{I} = 33/2$ // along the given path

ⓑ Equation of straight line joining (0, 3) & (2, 4) is given by

$$\frac{y-3}{x-0} = \frac{4-3}{2-0}$$

$$\frac{y-3}{x} = \frac{1}{2}$$

$$x = 2y-6$$

$$dx = 2dy$$

$$I = \int_{y=3}^{4} \{2y + \underset{(a-b)^2}{\underbrace{(2y-6)^2}}\}\, 2dy + \{3(2y-6)-y\}dy$$

$$= \int_3^4 \{2(2y + 4y^2 + 36 - 2\times 2y\times 6) + (6y-18-y)\}\, dy$$

$$= \int_3^4 \{(4y + 8y^2 + 72 - 48y) + (5y-18)\}\, dy \quad -44$$

$$= \int_3^4 (8y^2 - 39y + 54)\, dy$$

$$= \left[8\frac{y^3}{3} - 39\cdot\frac{y^2}{2} + 54y\right]_3^4$$

$$= \frac{8}{3}[4^3 - 3^3] - \frac{39}{2}[4^2 - 3^2] + 54[4-3]$$

$$= \frac{8}{3}[64-27] - \frac{39}{2}[16-9] + 54$$

$\underline{\underline{I = 97/6}}$ along the given path

⑤ Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the following paths:

ⓐ $x = t+1, \; y = 2t^2-1$

ⓑ straight line joining $(1-i)$ and $(2+i)$

(1

<u>Sol</u>: ⓐ $x = t+1, \; y = 2t^2-1$

$dx = dt, \; dy = 4t\,dt$

x vary from 1 to 2

If $x = 1, \; t = 0$

$x = 2, \; t = 1$

$dz = dx + i\,dy, \; dz = dt + i4t\,dt$

$$I = \int_{t=0}^{1} \{2(t+1) + i(2t^2-1) + 1\}\{dt + i4t\,dt\}$$

$$= \int_0^1 (2t + 2 + 1 + i2t^2 - i)\,dt\,(1 + i4t)$$

$$= \int_0^1 (2t + 2it^2 - i + 3)(1 + i4t)\,dt$$

$$= \int_0^1 \{ 2t + \underline{i8t^2} + \underline{2it^2} + i^2 8t^3 - i - i^2 4t + 3 + i12t \}\,dt$$

$$= \int_0^1 \{ \underline{2t} + i10t^2 - 8t^3 - i \underline{-(-1)4t} + 3 + i12t \}\,dt$$

$$= \int_0^1 \{ 2t + 4t + i10t^2 - 8t^3 - i + 3 + i12t \}\,dt$$

$$= \int_0^1 \{ 6t + i10t^2 - 8t^3 - i + 3 + i12t \}\,dt$$

$$\left[\frac{6t^2}{2} + i\,10\frac{t^3}{3} - 8\frac{t^4}{4} - it + 3t + i\,12\frac{t^2}{2}\right]_0^1$$

$$\left[3t^2 + i\frac{10}{3}t^3 - 2t^4 - it + 3t + i\,6t^2\right]_0^1$$

$$\left[\underline{3} + i\frac{10}{3} - \underline{2} - i + \underline{3} + i6\right]$$

$$= 4 + i\left(\frac{10}{3} - 1 + 6\right)$$

$$= 4 + i\left(\frac{10-3+18}{3}\right)$$

$I = 4 + i\frac{25}{3}$ along the given path.

Equation of the straight line joining $(1,-1)$ and $(2,1)$ is given by

$$\frac{y+1}{x-1} = \frac{1-(-1)}{2-1}$$

$$\frac{y+1}{x-1} = \frac{2}{1}$$

$$y+1 = 2x-2$$

$$y = 2x-2-1$$

$$y = 2x-3$$

$$dy = 2dx$$

$$I = \int_{x=1}^{2} \{2x + i(2x-3) + 1\}\{dx + i2dx\}$$

$$= \int_{x=1}^{2} \{2(1+i)x + (1-3i)\}(1+2i)dx$$

$$= \int_{x=1}^{2} \{2(1+i)(1+2i)x + (1-3i)(1+2i)\}\,dx$$

$$I = \int_{x=1}^{2} \{2(-1+3i)x + (1-3i)(1+2i)\}\,dx$$

$$= (1-3i)\int_{x=1}^{2} \{-2x + (1+2i)\}\,dx$$

$$= (1-3i)\left\{-\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_1^2 + (x+2ix)_1^2\right\}$$

$$= (1-3i)\{-(4-1) + (2+4i-1-2i)\}$$

$$= (1-3i)\{-3+1+2i\}$$

$$= (1-3i)(-2+2i)$$

$$= -2+2i+6i-6i^2$$

$$= -2+8i-6(-1)$$

$$= -2+8i+6$$

$I = \underline{\underline{4+8i}}$ along the given path dd

Cauchy's theorem :

statement : If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)\,dz = 0$

proof : Let $f(z) = u + iv$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$\int_C f(z)\,dz = \int_C (\underset{m}{u\,dx} - \underset{N}{v\,dy}) + i\int_C (\underset{m}{v\,dx} + \underset{N}{u\,dy}) \text{ — } ①$$

we have Green's theorem in a plane stating that if $M(x,y)$ & $N(x,y)$ are two real valued functions having Contineous first order partial derivatives in a region R bounded by curve C then

$$\int_C m\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy$$

$m = u$

$\frac{\partial m}{\partial y} = \frac{\partial u}{\partial y}$

$N = -v$

$\frac{\partial N}{\partial x} = -\frac{\partial v}{\partial x}$

$m = v$

$\frac{\partial m}{\partial y} = \frac{\partial v}{\partial y}$

$N = u$

$\frac{\partial N}{\partial x} = \frac{\partial u}{\partial x}$

Applying this theorem to the two lines integrals in the RHS of ① we obtain

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since f(z) is analytic we have Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & hence we have

$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = 0$$

This proves Cauchy's theorem.

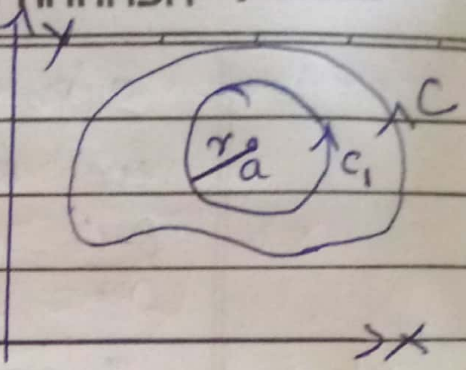
Cauchy's integral formula

Statement: If f(z) is analytic inside and on a simple closed curve C & if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$$

Proof: Since 'a' is a point within C, we shall enclose it by a circle C_1 with z = a as centre and r as radius r; C_1 lies entirely within C.

$$-\frac{\partial v}{\partial y}\Big)\,dx\,dy$$



The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region between C & C_1

Now as a consequence of Cauchy's theorem

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{— ①}$$

The equation of C_1 (Ole with centre 'a' & radius r) can be written in the form $|z-a| = r$. This is equivalent to $z-a = re^{i\theta}$

$$z = a + re^{i\theta} \quad 0 \le \theta \le 2\pi.$$

$$dz = ire^{i\theta}\,d\theta$$

use these results in the RHS of ①

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}}\, i\cancel{re^{i\theta}}\,d\theta$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r > 0$ however small. Hence as $r \to 0$ we get

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta$$

$$= i\,f(a)\int_{\theta=0}^{2\pi} 1.d\theta$$

$$= i f(a)\,[\theta]_0^{2\pi}$$

$$= i f(a)(2\pi - 0)$$

$$\int_C \frac{f(z)}{z-a}\,dz = i2\pi f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \;//$$

Cauchy's integral formula.

Generalized Cauchy's integral formula:

Statement: If $f(z)$ is analytic inside and on a simple closed curve C & if 'a' is a point within C then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof: we have Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Applying Leibnitz rule for differentiation under the integral sign we have

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\,\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right) dz$$

$$= \frac{1}{2\pi i}\int_C f(z) \times \frac{-1}{(z-a)^2} \times -1 \; dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz \quad - ②$$

again apply Leibnitz rule for ②

$$f''(a) = \frac{1!}{2\pi i}\int_C f(z)\,\frac{\partial}{\partial a}\left(\frac{1}{(z-a)^2}\right) dz$$

$$= \frac{1!}{2\pi i}\int_C f(z) \times \frac{-2}{(z-a)^3} \times -1 \; dz$$

$$= \frac{1!}{2\pi i}\int_C 2\,\frac{f(z)}{(z-a)^3}\, dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\, dz$$

Continuing like this after differentiating n times we obtain

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

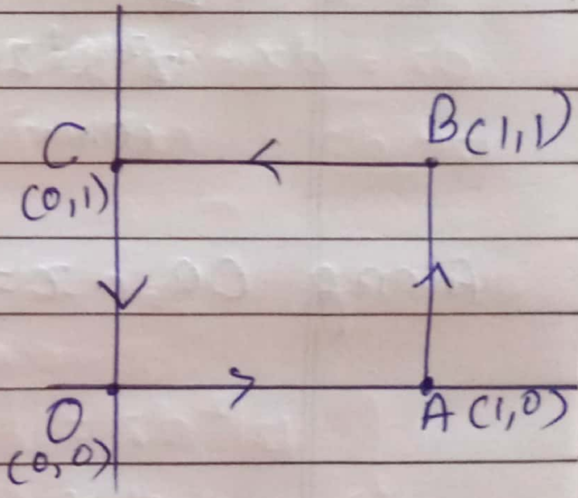
$f^{(n)}(a)$ denotes the n^{th} derivative of $f(z)$ at $z=a$.

Problems

① Verify Cauchy's theorem for the function $f(z)=z^2$ where C is the square having vertices (0,0) (1,0) (1,1) (0,1)

Sol^n C is the square OABC and we have by Cauchy's theorem

$$\int_C f(z)\, dz = 0$$



∴ we have to S.T

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{CO} z^2 dz = 0$$

Along OA, $y=0$, $dy=0$, $z^2 dz = (x+iy)^2\, dx + i dy$

$$z^2 dz = x^2 dx$$

where $0 \le x \le 1$

Along AB, $x=1$ $dx=0$, $z^2 dz = (x+iy)^2 dx + i dy$

$$z^2 dz = (1+iy)^2 i dy$$

$$z^2 dz = (1+i^2y^2+2iy) i dy$$

$$z^2 dz = (1-y^2+2iy) i dy$$

where $0 \le y \le 1$

Along BC, $y=1$, $dy=0$ $z^2 dz = (x+iy)^2 dx + i dy$

$$z^2 dz = (x+i)^2 dx$$

$$z^2 dz = (x^2+i^2+2xi) dx$$

$$z^2 dz = (x^2-1+2xi) dx$$

where $1 \le x \le 0$

Along CO, $x=0$, $dx=0$ $z^2 dz = (x+iy)^2 (dx+idy)$

$$z^2 dz = (iy)^2 (i dy)$$

$$z^2 dz = -y^2 i dy$$

where $1 \le y \le 0$

put all these in ①

① becomes

$$\int_0^1 x^2 dx + i\int_0^1 (1-y^2+2iy) dy + \int_1^0 (x^2-1+2xi) dx + \int_1^0 -y^2 i dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y - \frac{y^3}{3} + i\cancel{2}\frac{y^2}{\cancel{2}}\right]_0^1 + \left[\frac{x^3}{3} - x + i\cancel{2}\frac{x^2}{\cancel{2}}\right]_1^0 - i\left[\frac{y^3}{3}\right]_1^0$$

$$= \left(\frac{1}{3} - 0\right) + i\left(1 - \frac{1}{3} + i\right) + \left(0 - \left(\frac{1}{3} - 1 + i\right)\right)$$

$$- i\left(0 - \frac{1}{3}\right)$$

$$= \cancel{\frac{1}{3}} + \cancel{i} - \cancel{\frac{i}{3}} + i^2 \cancel{-\frac{1}{3}} + 1 \cancel{-i} \cancel{+\frac{i}{3}}$$

$$= -i^2 + 1$$

$$= -1 + 1$$

$$= 0$$

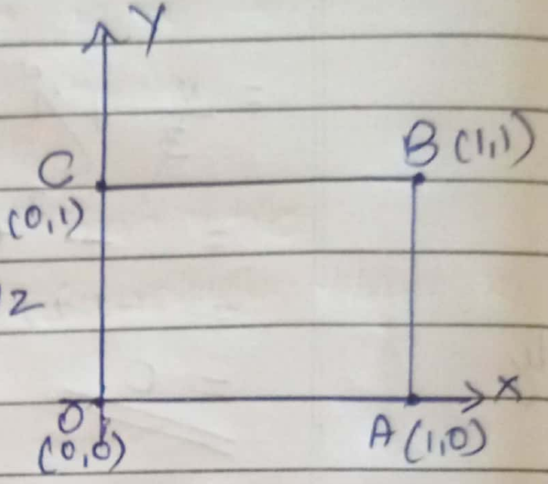
$$\int_C f(z)\,dz = 0$$

hence Cauchy's theorem is verified

(i+1) ② S.T $\int_C |z|^2 dz = i-1$ where C is the square having vertices (0,0) (1,0) (1,1) (0,1). Give reason for Cauchy's theorem not being satisfied.

Solⁿ

C is a square OABC



$$\int_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz \quad — ①$$

$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2$

$|z|^2 = x^2+y^2$

$|z|^2 = x^2+y^2$

$dz = dx + i\,dy$

$|z|^2 dz = (x^2+y^2)(dx + i\,dy)$

Along OA, $y=0,\ dy=0$ $|z|^2 dz = x^2 dx$, where $0 \le x \le 1$

$$\int_{OA} |z|^2 dz = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{1}{3} - 0\right) = \frac{1}{3}$$

Along AB, $x=1,\ dx=0$ $|z|^2 dz = (1+y^2)\,i\,dy$ where $0 \le y \le 1$

$$\int_{AB} |z|^2 dz = \int_0^1 (1+y^2)\,i\,dy = i\int_0^1 (1+y^2)\,dy = i\left[y+\frac{y^3}{3}\right.$$

$$= i\left[1+\frac{1}{3}-0\right]$$

$= i\,4/3$

Along BC, $y = 1$ $dy = 0$ $|z|^2 dz = (x^2+1)\,dx$
where $1 \leq x \leq 0$

$$\int_{BC} |z|^2 dz = \int_1^0 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_1^0 = \left[0 - (1/3 + 1)\right.$$

$= -4/3$ //

Along CO, $x = 0, dx = 0$ $|z|^2 dz = y^2\, i\,dy$
where $1 \leq y \leq 0$

$$\int_{CO} |z|^2 dz = \int_1^0 y^2\, i\,dy = i\left[y^3/3\right]_1^0 = i\left[0 - 1/3\right.$$

$= -i/3$ //

put all thyc in ①

$$\int_C |z|^2 dz = 1/3 + i\,4/3 - 4/3 - i/3$$

$= (1/3 - 4/3) + i\,(4/3 - 1/3)$

$= -3/3 + i\,3/3$

$\int_C |z|^2 dz = -1 + i$ // ⓐ $i - 1$ //

According to Cauchy's theorem we have to S.T $\int_C f(z)\,dz = 0$

here $f(z) = |z|^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2 \quad v = 0$

$u_x = 2x \quad v_x = 0$

$u_y = 2y \quad v_y = 0$

C-R eqn not satisfied

hence $f(z) = |z|^2$ is not analytic

This is the reason for Cauchy's theorem not being satisfied. //

③ Verify Cauchy's theorem for the function $f(z) = ze^{-z}$ over the unit ○'e with origin as the centre.

Soln: we have to evaluate $\int_C ze^{-z}\,dz$ where C is the ○'e $|z| = 1$

$z = re^{i\theta}, \quad 0 \le \theta \le 2\pi \quad dz = ie^{i\theta}d\theta$

$z = e^{i\theta}$

$$\int_C ze^{-z}\,dz = \int_{\theta=0}^{2\pi} e^{i\theta} e^{-e^{i\theta}} ie^{i\theta}\,d\theta$$

$$= i\int_{\theta=0}^{2\pi} e^{2i\theta} e^{-e^{i\theta}}\,d\theta$$

put $e^{i\theta} = t \quad \therefore e^{i\theta} i\,d\theta = dt$ or $d\theta = dt/it$

when $\theta = 0, \; t = e^0 = 1$

$\theta = 2\pi, \; t = e^{2\pi i} = \cos 2\pi + i\sin 2\pi$

$t = 1$

$$\int_C ze^{-z}dz = \cancel{i}\int_{t=1}^{1} \cancel{t}^2 e^{-t}\frac{dt}{i\cancel{t}}$$

$$= \int_{t=1}^{1} t e^{-t}\,dt$$

Since the limits are same, the value of the integral is zero.

$\int_C ze^{-z} = 0$ Cauchy's theorem is verified.

Problems on Cauchy's Integral formula

working procedure:-

① we need to evaluate integrals of the form $\int_C \frac{f(z)}{z-a}dz$; $\int_C \frac{f(z)}{(z-a)^{n+1}}dz$ over a given closed curve C.

② firstly we have to find out whether the point $z=a$ lies inside or outside the given curve C.

③ If $z=a$ is inside C then we use Cauchy's integral formula in the forms

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a),\quad \int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a).$$

④ If the point $z=a$ is outside C and supposing that $F(z) = \frac{f(z)}{z-a}$ (or) $\frac{f(z)}{(z-a)^{n+1}}$ is analytic inside and on the given curve C we can conclude that $\int_C f(z)dz = 0$ by Cauchy's theorem

* In other words if $z=a$ is outside C the value of integral is zero

problems

① Evaluate $\int_C \frac{e^z}{z+i\pi} dz$ over each of the following contours C:

ⓐ $|z| = 2\pi$ ⓑ $|z| = \pi/2$ ⓒ $|z-1| = 1$

Sol^n:

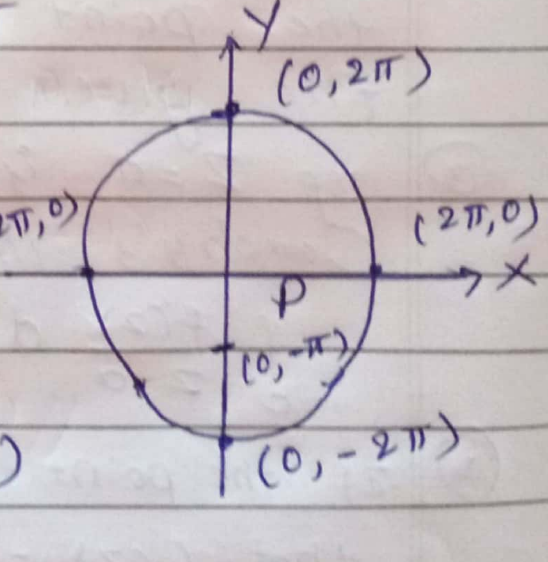
$$\int_C \frac{e^z}{z+i\pi} dz$$

$\int_C \frac{e^z}{z-(-i\pi)} dz$ is of the form $\int_C \frac{f(z)}{z-a} dz$

here $f(z) = e^z$ $a = -i\pi$

ⓐ $|z| = 2\pi$ is a circle with centre $(0,0)$ and radius 2π

The point $z = a = -i\pi$ is the point $P(0,-\pi)$



liy within the Circle $|z| = 2\pi$

we have C.I.F

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

w.K.T $f(z)=e^z$, $a=-i\pi$

$$\int_C \frac{e^z}{z+i\pi}\,dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi}$$

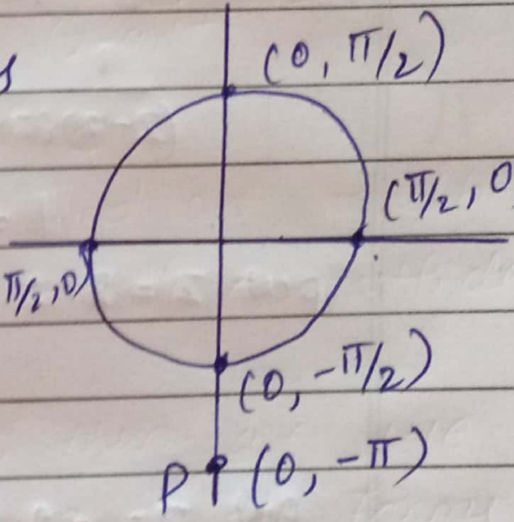
$$= 2\pi i(\underset{(-1)}{\cos\pi} - i\underset{0}{\sin\pi})$$

$$\int_C \frac{e^z}{z+i\pi}\,dz = -2\pi i$$ //

(b) $|z| = \pi/2$

is a circle with Centre origin and radius $\pi/2$. The point $(0,-\pi)$ liy out side the ○le $|z|=\pi/2$ and $\frac{e^z}{z+i\pi}$ is analytic inside and on the circle $|z|=\pi/2$. By Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\,dz = 0$$

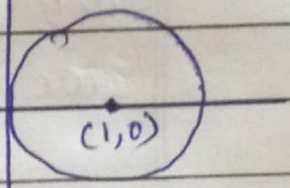


(c) $|z-1| = 1$ is a ○le with Centre at $z=a=1$ and radius 1. This is a ○le with Centre $(1,0)$ & radius 1.

The point $P(0,-\pi)$ lies outside the $\bigcirc^{le}$

$\therefore$ by Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\,dz = 0$$



② Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curves C.

ⓐ $C: |z|=1$ ⓑ $C: |z|=3$ ⓒ $C: |z+2|=1$

Soln: Consider $\frac{1}{z^2-4} = \frac{1}{z^2-2^2} = \frac{1}{(z-2)(z+2)}$

Resolving into partial fractions

$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z-2)$$

put $z=2$: $1 = 4A + 0$

$1 = 4A \Rightarrow A = 1/4$

put $z=-2$: $1 = 0 + B(-4)$

$1 = -4B$

$B = -1/4$ //

$$\frac{1}{(z-2)(z+2)} = \frac{1}{4(z-2)} - \frac{1}{4(z+2)}$$

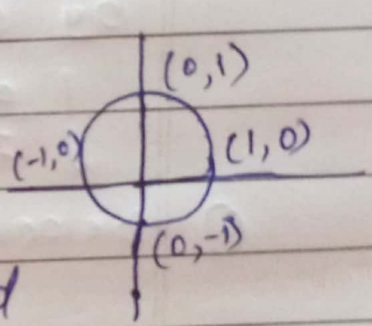
$$\int \frac{1}{(z-2)(z+2)} dz = \frac{1}{4}\int \frac{1}{z-2} dz - \frac{1}{4}\int \frac{1}{z-(-2)} dz$$

— ①

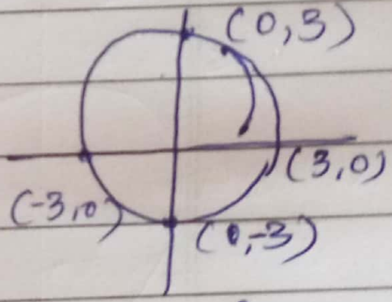
ⓐ $C : |z| = 1$; $z = a = 2$ and $z = a = -2$ lies outside the circle.

∴ by Cauchy's theorem

$\int_C \frac{dz}{z^2-4} = 0$ where $C : |z| = 1$ //



ⓑ $C : |z| = 3$; $z = a = 2$ and $z = a = -2$ lies inside the circle



W.K.T here $f(z) = 1$
from eqⁿ ①

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{f(z)}{z-2} dz = 2\pi i f(2) = 2\pi i(1) = 2\pi i$$

$$\int_C \frac{f(z)}{z+2} dz = 2\pi i f(-2) = 2\pi i(1) = 2\pi i$$

put these in ①

$$\int_C \frac{dz}{(z-2)(z+2)} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0 //$$

$$\int_C \frac{dz}{z^2-4} dz = 0 //$$

ⓒ $C : |z+2| = 1$

This is a $\bigcirc^{le}$ with centre $(-2, 0)$ and radius 1

The point $(2,0)$ lies outside the $\bigcirc^{le}$ hence by Cauchy's theorem

$$\int \frac{dz}{z-2} = 0$$

The point $(-2, 0)$ lies inside the $\bigcirc^{le}$ by Cauchy integral formula

$$\int \frac{dz}{z+2} = \int \frac{dz}{z-(-2)} = 2\pi i f(-2) = 2\pi i(1) = 2\pi i$$

① becomes $\displaystyle\int \frac{dz}{z^2+4} = \frac{1}{4}(0) - \frac{1}{4}(2\pi i)$

$$= -\frac{\pi i}{2} /\!/$$

2 //

③ Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)}\, dz$ where C is the O^{le} $|z| = 3$

Sol^n: $\int \frac{e^{2z}}{(z+1)(z-2)}\, dz$

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$ — ⊛

$1 = A(z-2) + B(z+1)$ — ①

put $z = 2$ in ①

$1 = 0 + B(3)$

$3B = 1 \Rightarrow B = 1/3$

put $z = -1$ in ①

$1 = A(-1-2) + 0$

$1 = -3A$

$A = -1/3$

⊛ becomes

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = -\frac{1}{3}\int \frac{1}{z+1} dz + \frac{1}{3}\frac{1}{z-2}$$

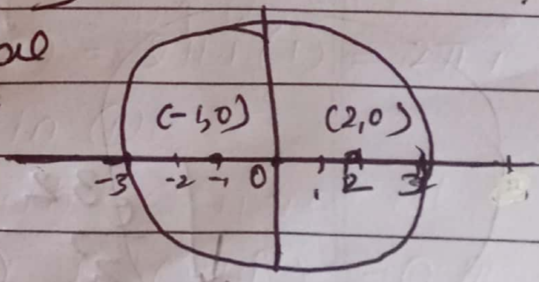
$$\frac{1}{(z+1)(z-2)} = \frac{1}{3}\frac{1}{z-2} - \frac{1}{3(z+1)}$$

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{1}{3}\int \frac{e^{2z}}{z-2} dz - \frac{1}{3}\int \frac{e^{2z}}{\underset{\hookrightarrow z-(-1)}{z+1}} dz$$

$|z| = 3$ is a O^{le} with centre $(0,0)$ & radius 3.

The point $z = a = 2$, and the point $z = a = -1$ both lie inside the O^{le}.

hence by Cauchy integral formula, w.k.t $f(z) = e^{2z}$



$$\int \frac{e^{2z}}{z-2} dz = 2\pi i f(2)$$
$$= 2\pi i e^{2\times 2}$$
$$= 2\pi i e^{4}$$

No need
Just
refrence

$$\int \frac{e^{2z}}{z+1} dz = \int \frac{e^{2z}}{z-(-1)} dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$$

put in ①

① becomes

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{2\pi i e^4}{3} - \frac{2\pi i}{3e^2}$$

$$= \frac{2\pi i}{3}\left(e^4 - 1/e^2\right)$$//

(ii) Evaluate $\int \frac{e^{3z}}{z^2} dz$ over $c: |z| = 1$

Solⁿ: $|z|=1$ is a circle with the Centre $(0,0)$ and radius 1. The point $z=0$ lies inside the circle & we have by Cauchy integral formula in the generalized form

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Take $f(z) = e^{3z}$, $a=0$, $n=1$

$$\int \frac{e^{3z}}{(z-0)^{1+1}} dz = \int \frac{e^{3z}}{z^2} dz = \frac{2\pi i}{1!} f'(a)$$

$$f'(z) = 3e^{3z}, \quad z=a=0$$

$$f'(0) = 3e^0 = 3$$

$$\int \frac{e^{3z}}{z^2} dz = 2\pi i(3) = 6\pi i$$//

⑤ Evaluate $\int \frac{z^2+z+1}{(z-2)^3}\,dz$ over $C: |z|=3$

Sol^n: The $|z|=3$ is a circle with centre $(0,0)$ & radius 3. The point $z=2$ lies inside the $\bigcirc^{le}$. $\therefore$ by Cauchy's integral formula

$$\int \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = z^2+z+1$

$$f'(z) = 2z+1$$
$$f''(z) = 2$$

$$\int \frac{z^2+z+1}{(z-2)^3}\,dz = \int \frac{z^2+z+1}{(z-2)^{\overset{2+1}{n+1}}}\,dz = \frac{2\pi i}{2!} f^{(2)}(a)$$

$$\downarrow$$
$$f''(a)$$

$$z = a = 2$$

$$f''(z) = f''(2) = 2$$

$$\int \frac{z^2+z+1}{(z-2)^3}\,dz = \frac{2\pi i}{2!} f''(2) = \frac{2\pi i}{\cancel{2}} \times \cancel{2} = 2\pi i \,//$$

⑥ Evaluate $\int \frac{e^{\pi z}}{(2z-i)^3}\,dz$ where C is $\bigcirc^{le}$

$|z|=1$

Sol^n: $|z|=1$ is a $\bigcirc^{le}$ with centre $(0,0)$ & radius is 1.

$$\int \frac{e^{\pi z}}{(2z-i)^3}\,dz = \int \frac{e^{\pi z}}{\left(2\left(z-\frac{i}{2}\right)\right)^{2+1}} \longrightarrow \text{express in terms of } (z-a)^{n+1}$$

The point $z = a = i/2$ i.e $(0, 1/2)$ lies inside the o'e

by Cauchy integral Generalized formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \int \frac{e^{\pi z}}{2^3(z-i/2)^{2+1}}\,dz = \frac{1}{8}\times\frac{2\pi i}{2!}f''(a)$$

$$f(z) = e^{\pi z}$$

$$f'(z) = \pi e^{\pi z}$$

$$f''(z) = \pi^2 e^{\pi z}$$

$$z = a = -i/2$$

$$f''(-i/2) = \pi^2 e^{-i\pi/2}$$

$$\circledast \Rightarrow \frac{1}{8}\int \frac{e^{\pi z}}{(z-i/2)^3} = \frac{1}{8}\times\frac{\cancel{2}\pi i}{\cancel{2}!}f''(a)$$

$$= \frac{1}{8}\pi i f''(-i/2)$$

$$= \frac{1}{8}\pi i \cdot \pi^2 e^{-i\pi/2}$$

$$= \frac{1}{8} i\pi^3 e^{-i\pi/2}$$

$$= \frac{1}{8} i\pi^3 (\cos\pi/2 - i\sin\pi/2)$$

$$= \frac{1}{8} i^3\pi^3 (0 - i(1))$$

$$= \frac{1}{8}\times - i^4\pi^3$$

$$= -\pi^3/8$$

⑦ Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz$ where $C: |z| = 3$

— ✳

Sol

Sol^n: we shall first resolve $\frac{1}{(z+1)^2(z-2)}$ into partial fractions.

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$$

$$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

put $z = -1$

$1 = 0 + B(-3) + 0$

$-3B = 1 \Rightarrow B = -1/3$

put $z = 2$

$1 = A(0) + 0 + C(3)^2$

$1 = 9C \Rightarrow C = 1/9$

put $z = 0$

$1 = A(1)(-2) + B(-2) + C(1)$

$1 = -2A + 2/3 + 1/9$

~~$2A = 2/3 + 1/9 = 1$~~ $\quad 3\,\underline{|\,9 \cdot 6}$ $\quad 3 \; 2$

$2A = \frac{\frac{1}{3}\times\frac{18}{1} + \frac{1}{9}\times\frac{18}{1} - \frac{18}{1}}{\frac{18}{1}} = \frac{3+1+9}{9}$ 1F

$= \frac{3+2-18}{18}$

$2A = \frac{-12}{18} \Rightarrow A = \frac{-12}{18\times 2} \Rightarrow A = -$

$1 = 2A + 2/3 + 1/9$

$$2A = 2/3 + 1/9 - 1$$

$$= \frac{6+1-9}{9}$$

$$\cancel{2}A = -\cancel{2}/9$$

$$\underline{\underline{A = -1/9}}$$

$$\frac{1}{(z+1)^2(z-2)} = \frac{-1}{9} \times \frac{1}{z+1} - \frac{1}{3} \times \frac{1}{(z+1)^2} + \frac{1}{9} \times \frac{1}{z-2}$$

$\times^{ly}$ e^{2z} & Integrate w.r.to z over C

$$\int \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{-1}{9} \int \frac{e^{2z}}{z+1} dz - \frac{1}{3} \int \frac{e^{2z}}{(z+1)^2} dz + \frac{1}{9} \int \frac{e^{2z}}{z-2} dz$$

$$\underline{I} = I_1 + I_2 + I_3 \quad - ⊛$$

$$I_1 = \frac{-1}{9} \int \frac{e^{2z}}{z+1} dz = \frac{-1}{9} \int \frac{e^{2z}}{z-(-1)} dz$$

The point $z = a = -1$ i.e $(-1, 0)$ lies inside the circle $\therefore \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

W.K.T $f(z) = e^{2z}$, $f($

$$f(-1) = e^{-2}$$

$$-\frac{1}{9} \int \frac{e^{2z}}{z+1} = 2\pi i f(-1) \times \frac{-1}{9}$$

$$= 2\pi i e^{-2} \times \frac{-1}{9}$$

$$= -\frac{2\pi i e^{-2}}{9}$$

$I_2 = \frac{-1}{3}\int \frac{e^{2z}}{(z+1)^2}\,dz$

$\Rightarrow \int \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i f^{(n)}(a)}{n!}$

$-\frac{1}{3}\int \frac{e^{2z}}{(z-(-1))^{1+1}}\,dz = \frac{-1}{3} \times \frac{2\pi i f'(a)}{1!}$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$z = a = -1$

$f'(-1) = 2e^{-2}$

dz

$-\frac{1}{3}\int \frac{e^{2z}}{(z+1)^2}\,dz = \frac{-1}{3} \times 2\pi i \times 2e^{-2}$

$= \frac{-4\pi i e^{-2}}{3}$

$I_3 = \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz$

$z = a = 2$ lies inside the O^{le}

$\int \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$

$= \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz = \frac{1}{9} \times 2\pi i f(2)$

w.k.t $f(z) = e^{2z}$

$f(2) = e^{2\times 2} = e^4$

$I_3 = \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz = \frac{1}{9} \times 2\pi i \times e^4$

Now $\circledast \Rightarrow$

$$\underline{I} = \frac{-2\pi i e^{-2}}{9} - \frac{4\pi i e^{-2}}{3} + \frac{2\pi i}{9} e^{4}$$

$$\underline{I} = \frac{2\pi i}{9}\left(\frac{-1}{9e^{2}} - \frac{2}{3e^{2}} + \frac{e^{4}}{9}\right)$$

module-03

Probability distributions

Let E be the event then the probability of an event 'E' is defined as

$$P(E) = \frac{\text{No. of favourable cases}}{\text{No. of possible cases}}$$

Random variable: In a random experiment if a real variable is associated with every outcome then it is called random variable.

ex①: while tossing a coin, suppose that the value 1 is associated for the outcome 'head' and 0 for the outcome 'tail'. we have the sample space $S = \{H, T\}$ and if X is the random variable then $X(H) = 1$ and $X(T) = 0$

Range of $X = \{0, 1\}$

ex②: Suppose a coin is tossed twice we shall associate two different random variables X, Y as follows.

we have sample space $S = \{HH, HT, TH, TT\}$

X - No. of heads in the outcomes

The association of elements in S to X as follows

outcome	HH	HT	TH	TT
Random variable X	2	1	1	0

Range of $X = \{0, 1, 2\}$

Suppose Y = No. of tails in the outcome

outcome	HH	HT	TH	TT
Random variable X	0	1	1	2

Range of $Y = \{0, 1, 2\}$

Types of Random Variable :-

① Discrete random Variable
② Contineous random Variable

① Discrete random Variable : If a random Variable takes finite no. of values (or) countably infinite no. of values then it is called discrete random Variable

ex : ① Tossing a coin and observing the outcome
② Tossing coins and observing the number of heads turning up.
③ Throwing a die & observing the numbers on the face.

② Contineous random Variable :-
If a random Variable takes infinite number of Values then it is called Contineous random Variable

ex ①: weight of articles

② Length of nails produced by machine

Probability function :-

If for each value of x_i of a discrete random variable 'X', we assign a real number $p(x_i)$ ∋;

① $p(x_i) \geq 0$

② $\sum p(x_i) = 1$

Mean and variance for general frequency distribution

$$(\text{Mean})\ \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$\text{Variance } (\sigma^2) = \frac{\sum (x_i - \bar{x})^2 f_i}{\sum f_i}$$

Mean and variance for discrete probability distribution :-

$$\text{mean } (\mu) = \sum x_i p(x_i)$$

$$\text{Variance } (V) = \sum (x_i - \mu)^2 p(x_i)$$

(or)

$$\text{Variance } (V) = \sum x_i^2 p(x_i) - \mu^2$$

$$\text{standard deviation } (\sigma) = \sqrt{\text{variance}}$$

NOTE①: $p+q=1$

where $p \to$ probability of Success
$q \to$ probability of failure

NOTE②: $P(A)+P(\bar{A})=1$

$P(A)=1-P(\bar{A})$

where $\bar{A}$ is the Complement of A.

Problems

① S.T the following distribution represents a discrete probability distribution, find the mean and Variance

x	10	20	30	40
$p(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Solⁿ: To prove the discrete probability distribution we have to verify the two conditions

① $p(x) \geq 0$

② $\Sigma p(x)=1$

ⓘ $p(x) \geq 0$, first Condition is satisfied

ⓘⓘ $\frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8} = \frac{1+3+3+1}{8} = \frac{8}{8}$

$=1$

$\therefore \Sigma P(x)=1$

we observed that both the Conditions are satisfied

$\therefore$ given distribution represents a discrete probability distribution mean $(\mu) = \sum x_i \, P(x_i)$

$= 10(1/8) + 20(3/8) + 30(3/8) + 40(1/8)$

$= \frac{10}{8} + \frac{60}{8} + \frac{90}{8} + \frac{40}{8}$

$= \frac{200}{8}$

$\mu = 25$

Variance (V) $= \sum (x_i - \mu)^2 P(x_i)$

$= (10-25)^2 \frac{1}{8} + (20-25)^2 \frac{3}{8} + (30-25)^2 \frac{3}{8} + (40-25)^2 \frac{1}{8}$

$= \frac{225}{8} + 25 \times \frac{3}{8} + 25 \times \frac{3}{8} + 225 \times \frac{1}{8}$

$= \frac{225 + 75 + 75 + 225}{8}$

$= \frac{600}{8}$

$= 75$

S.D $(\sigma) = \sqrt{V}$

$= \sqrt{75}$

② Find the Value of K Such that the following distribution represents a finite probability distribution and also find $P(x \le 1)$, $P(x > 1)$ and $P(-1 < x \le 2)$

x	-3	-2	-1	0	1	2	3
$p(x)$	K	$2K$	$3K$	$4K$	$3K$	$2K$	K

Solⁿ: we must have ⓘ $p(x) \geq 0$

ⓘⓘ $\sum P(x) = 1$

The first Condition is satisfied if $K \geq 0$, we have to find $K \ni; \sum P(x) = 1$

$$\sum P(x) = K + 2K + 3K + 4K + 3K + 2K + K = 16K$$

$$\sum P(x) = 1$$

$$\Rightarrow 16K = 1$$

$$K = \frac{1}{16},$$

$$\therefore K = \frac{1}{16} > 0$$

$$u0-25)^2 \ \frac{1}{8}$$

The discrete / finite probability distribution is

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

mean $\mu = \sum x_i P(x_i)$

$$= \frac{1}{16}(-3) + \frac{2}{16}(-2) + \frac{3}{16}(-1) + \frac{4}{16}(0) +$$

$$\frac{3}{16}(1) + \frac{2}{16}(2) + \frac{1}{16}(3)$$

$$= \frac{1}{16}(\cancel{-3} - \cancel{4} - \cancel{3} + 0 + \cancel{3} + \cancel{4} + \cancel{3})$$

$$= \frac{1}{16}(0)$$

$$\boxed{\mu = 0}$$

Variance $V = \sum (x_i - \mu)^2 p(x_i)$

$$V = (-3-0)^2 \frac{1}{16} + (-2-0)^2 \frac{2}{16} + (-1-0)^2 \frac{3}{16}$$

$$+ (0-0)^2 \frac{4}{16} + (1-0)^2 \frac{3}{16} + (2-0)^2 \frac{2}{16}$$

$$+ (3-0)^2 \frac{1}{16}$$

$$= 9 \times \frac{1}{16} + 4 \times \frac{2}{16} + 1 \times \frac{3}{16} + 0 \times \frac{4}{16} + 1 \times \frac{3}{16}$$

$$+ 4 \times \frac{2}{16} + 9 \times \frac{1}{16}$$

$$= \frac{1}{16}(9 + 8 + 3 + 0 + 3 + 8 + 9)$$

$$= \frac{11}{16}(40)$$

$V = \frac{5}{2}$ // $\quad S.D(\sigma) = \sqrt{V} = \sqrt{5/2}$

Thus $K = \frac{1}{16}$, $\mu = 0$, $V = \frac{5}{2}$, $\sigma = \sqrt{5/2}$

now we have to find $p(x) \leq 1$

$$P(x \leq 1) = P(-3) + P(-2) + P(-1) + P(0) + P(1)$$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$= \frac{1}{16}(1 + 2 + 3 + 4 + 3)$$

$P(x \leq 1) = \frac{13}{16}$ //

(ii) $P(x>1) = P(2)+P(3)$

$$= \frac{2}{16} + \frac{1}{16}$$

$$= \frac{1}{16}(2+1)$$

$$= \frac{3}{16} //$$

(iii) $P(-1 < x \leq 2) = P(0)+P(1)+P(2)$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$= \frac{4+3+2}{16}$$

$$P(-1 < x \leq 2) = \frac{9}{16} //$$

③ The probability distribution function of a variate X is given by the following table

x	0	1	2	3	4	5	6
$p(x)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

For what value of K this represents a valid probability distribution? also find $P(x \geq 5)$ and $P(3 < x \leq 6)$

Solⁿ: The Probability distribution is Valid if ① $P(x) \geq 0$ and

② $\Sigma P(x) = 1$

hence we must have $K \geq 0$ and

$$\Sigma P(x) = 1$$

$$K + 3K + 5K + 7K + 9K + 11K + 13K = 1$$

$$49K = 1$$

$$K = \frac{1}{49}$$

∴ probability distribution function is valid

for $K = \frac{1}{49}$

Now we have to find $p(x \geq 5)$ & $P(3 < x \leq 6)$

$$P(x \geq 5) = P(5) + P(6)$$
$$= 11K + 13K$$
$$= 24K$$
$$= 24 \times \frac{1}{49}$$
$$= \frac{24}{49}$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$
$$= 9K + 11K + 13K$$
$$= 33K$$
$$= 33 \times \frac{1}{49}$$
$$P(3 < x \leq 6) = \frac{33}{49}$$

④ The probability distribution of a finite random variable X is given by the following table

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	$2K$	0.3	K

Find the value of K, mean & Variance

Solⁿ: we must have $p(x) \geq 0$ and $\sum p(x) = 1$ for a probability distribution

we have to find K ∋;

$$\sum P(x) = 1$$
$$0.1 + K + 0.2 + 2K + 0.3 + K = 1$$
$$4K + 0.6 = 1$$

$4K = 1 - 0.6$

$4K = 0.4$

$K = 0.1$ //

The probability distribution of a finite random variable X is

x	-2	-1	0	1	2	3
$p(x)$	0.1	0.1	0.2	0.2	0.3	0.1

mean $\mu = \sum x_i p(x_i)$

$= -2(0.1) + (-1)(0.1) + 0(0.2) + 1(0.2) + 2(0.3) + 3(0.1)$

$= \cancel{-0.2} - 0.1 + 0 + \cancel{0.2} + 0.6 + 0.3$

$\mu = 0.8$

Variance $(v) = \sum (x_i - \mu)^2 p(x_i)$

$= (-2-0.8)^2(0.1) + (-1-0.8)^2(0.1) + (0-0.8)^2(0.2) + (1-0.8)^2(0.2) + (2-0.8)^2(0.3) + (3-0.8)^2(0.1)$

$= (-2.8)^2(0.1) + (-1.8)^2(0.1) + (-0.8)^2(0.2) + (0.2)^2(0.2) + (1.2)^2(0.3) + (2.2)^2(0.1)$

$= 7.84 \times 0.1 + 3.24 \times 0.1 + 0.64 \times 0.2 + 0.04 \times 0.2 + 1.44 \times 0.2 + 4.84 \times 0.1$

$= 0.784 + 0.324 + 0.128 + 0.008 + 0.432 + 0.484$

$V = 2.16$

⑤ A random Variable X has the following probability function for various values of x

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2k$	$3K$	K^2	$2K^2$	$7K^2+K$

(i) Find K

(ii) Evaluate $P(x<6)$, $P(x \geq 6)$ and $p(3<x\leq 6)$

also find the probability distribution and distribution function of x.

Solⁿ: To find the probability distribution we must have $p(x)\geq 0$ and $\Sigma p(x)=1$

The first Condition is satisfied for $K\geq 0$, we have to find K ∋;

$\Sigma p(x) = 1$

$$0+K+2k+2k+3K+K^2+2K^2+7K^2+K=1$$

$$10K^2+9K-1=0$$

$$10K^2+10K-K-1=0$$

$$10K(K+1)-1(K+1)=0$$

$$(K+1)(10K+1)=0$$

$K+1=0$ (or) $10K-1=0$

$K=-1$ (or) $10K=1$

$$K=\frac{1}{10}$$

If $K=-1$, first condition fails

So $K\neq -1$

∴ Take $K=\frac{1}{10}=0.1$

$$\underline{K=0.1}$$

hence table (or) probability distribution is

x	0	1	2	3	4	5	6	7
$p(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(ii) $p(x < 6)$

$= P(0) + P(1) + P(2) + P(3) + P(4) + P(5$

$= 0 + 0.1 + 0.2 + 0.2 + 0.3 + 0.01$

$p(x<6) = 0.81$

$p(x \geq 6) = p(6) + p(7)$

$= 0.02 + 0.17$

$= 0.19$

$p(3 < x \leq 6) = P(4) + P(5) + P(6)$

$= 0.3 + 0.01 + 0.02$

$= 0.33$

distribution function of x is

x	0	1	2	3	4	5	6	7
$f(x)$	0	$0+0.1$ $=0.1$	$0.1+0.2$ $=0.3$	$0.3+0.2$ $=0.5$	$0.5+0.3$ $=0.8$	$0.8+0.0$ $=0.81$	$0.81+$ $0.02=0.83$	$0.83+0.17$ $=1$

⑥ A random variable x take the values $-3, -2, -1, 0, 1, 2, 3$ such that $p(X=0)=p(X<0)$ and $p(X=-3) = p(X=-2) = p(X=-1) = p(X=1) = p(X=2) = p(X=3)$. find the probability distribution.

Soln :

X	-3	-2	-1	0	1	2	3
p(x)	P_1	P_2	P_3	P_4	P_5	P_6	P_7

By data $p(x=0) = p(x<0)$

$\Rightarrow P(X=0) = p(X=-1) + P(X=-2) + p(X=-3)$

$$P_4 = P_3 + P_2 + P_1 \quad \text{— ①}$$

By data

$$P_1 = P_2 = P_3 = P_5 = P_6 = P_7 \quad \text{— ②}$$

We must have $\Sigma P(x) = 1$

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1 \quad \text{— ③}$$

use ② in ③

$$P_1 + P_1 + P_1 + P_4 + P_1 + P_1 + P_1 = 1$$

$$6P_1 + P_4 = 1 \quad \text{— ④}$$

use ③ in ①

eqⁿ ① becomes $P_4 = P_1 + P_1 + P_1$

$P_4 = 3P_1$

eqⁿ ④ becomes

$6P_1 + 3P_1 = 1$

$9P_1 = 1$

$\boxed{P_1 = \frac{1}{9}}$

hence $P_4 = 3P_1 = 3 \times \frac{1}{9} = \frac{1}{3}$

$\therefore \boxed{P_4 = \frac{1}{3}}$

Thus probability distribution is

X	-3	-2	-1	0	1	2	3
P(X)	1/9	1/9	1/9	1/3	1/9	1/9	1/9

⑦ If the random variable X take the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution function of X.

Solⁿ

X	1	2	3	4
P(X)	P_1	P_2	P_3	P_4

$2P_1 = 3P_2 = P_3 = 5P_4$ — ①

also we must have $\sum P(x) = 1$

$P_1 + P_2 + P_3 + P_4 = 1$ — ②

from ①

$2P_1 = 3P_2$, $2P_1 = P_3$, $2P_1 = 5P_4$

$P_2 = \frac{2}{3}P_1$ $\quad P_3 = 2P_1$ $\quad P_4 = \frac{2}{5}P_1$

hence ② becomes

$$P_1 + \frac{2}{3}P_1 + 2P_1 + \frac{2}{5}P_1 = 1$$

lcm is 15

$$\frac{15P_1 + 10P_1 + 30P_1 + 6P_1}{15} = 1$$

$$\frac{61}{15}P_1 = 1 \Rightarrow P_1 = \frac{15}{61}$$

hence we get $P_2 = \frac{2}{3} \times P_1 = \frac{2}{3} \times \frac{15}{61} = \frac{\overset{10}{\cancel{30}}}{\cancel{3} \times 61} = \frac{10}{61}$

$P_3 = 2P_1 = 2 \times \frac{15}{61} = \frac{30}{61}$ //

$P_4 = \frac{2}{5}P_1 = \frac{2}{5} \times \frac{15}{61} = \frac{6}{61}$ //

The probability distribution function of X

X	1	2	3	4
P(X)	$15/61$	$10/61$	$30/61$	$6/61$
f(x)	$15/61$	$15/61 + 10/61 = 25/61$	$25/61 + 30/61 = 55/61$	$55/61 + 6/61 = 61/61 = 1$

⑧ If X is a discrete random variable having p(x) defined as follows

$$p(x) = \begin{cases} x/15, & \text{if } 1 \le x \le 5 \\ 0 & \text{if } x > 5 \end{cases}$$

S.T p(x) is a probability function and find p(X=1 or 2)

Solⁿ: The probability distribution is as follows

x	1	2	3	4	5	6, 7, 8....
$p(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$	0

we have ① $p(x) \geq 0$

② $\sum p(x) = 1$

$\therefore$ $p(x)$ is probability function

Now $P(X = 1 \text{ or } 2) = P(X=1) + P(X=2)$

$$= \frac{1}{15} + \frac{2}{15}$$

$$= \frac{3}{15}$$

$$= \frac{1}{5}$$

Binomial distribution

If p is the probability of success and q is the probability of failure, the probability of x success out of n trials is given by $p(x) = {}^nC_x\, p^x q^{n-x}$

we form the following probability distribution of $(x, p(x))$ where $x = 0, 1, 2 \cdots n$

x	0	1	2	n
$p(x)$	q^n	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$	p^n

It may be observed that the value of $p(x)$ for different values $x = 0, 1, 2 \cdots n$ are the successive terms in the binomial expansion of $(q+p)^n$ and accordingly this distribution is called binomial distribution.

$$\Sigma p(x) = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \cdots + p^n$$
$$= (q+p)^n$$
$$= 1^n$$
$$= 1$$

hence $p(x)$ is a probability function

Mean & Standard deviation of Binomial distribution :-

(OR)

Derive Mean & S.D of Binomial distribution

W.K.T

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

$$\text{Mean, } \mu = \sum_{x=0}^{n} x\,p(x)$$

$$= \sum_{x=0}^{n} x\,{}^nC_x\,p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x\,\frac{n!}{(n-x)!\,x!}\,p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x}\,n(n-1)!}{(\cancel{x})(x-1)!\,(n-x)!}\,p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!}\,p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{x=\phi}^{n} \frac{(n-1)!}{(x-1)!\,\left[(n-1)-(x-1)\right]!}\,p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{x-1}\,p^{x-1} q^{(n-1)-(x-1)}$$

$$\frac{n!}{r!\,(n-r)!} = {}^nC_r$$

$$= np\,(q+p)^{n-1}$$

$$= np\,(1)$$

$$\mu = np \,//$$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 p(x) - \mu^2 \quad \text{— } ⊛$$

$$\text{Now, } \sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} \left[x(x-1) + x\right] p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\,p(x) + x\,p(x)$$

$= \sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=0}^{n} xp(x)$

$= \sum_{x=0}^{n} x(x-1)\, n_{C_x}\, p^x q^{n-x} + np$

$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^x q^{n-x} + np$

$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n(n-1)(n-2)!}{(n-x)!\, \cancel{x(x-1)}\, [x-2]!} p^2 \cdot p^{x-2} q^{n-x} + np$

$= \sum_{x=0}^{n} \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^2 p^{x-2} q^{n-x} + np$

$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,[(n-2)-(x-2)]!} p^{x-2} \cdot q^{(n-2)-(x-2)} + np$

$= n(n-1)p^2 \sum_{x=2}^{n} (n-2)_{C_{(x-2)}}\, p^{x-2} q^{(n-2)-(x-2)} + np$

$= n(n-1)p^2 (q+p)^{n-2} + np$

$= n(n-1)p^2 (1)^{n-2} + np$

$\sum x^2 p(x) = n(n-1)p^2 + np$

$= (n^2-n)p^2 + np$

$\sum x^2 p(x) = n^2p^2 - np^2 + np$ //

⊛ becomes

∴ Variance V

=

=

variance (V

S.D (σ) =

Thus for

$\mu = np$,

① Find

which

Soln: w.k

P

(*) becomes

∴ Variance $V = (n^2p^2 - np^2 + np) - (np)^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$

$= -np^2 + np$

$= np(-p+1)$

$= np(1-p)$

$= npq$

Variance $(V) = npq$

S.D $(\sigma) = \sqrt{V} = \sqrt{npq}$

W.K.T
$p+q=1$
$q = 1-p$

+np

Thus for binomial distribution

$\mu = np$, Variance $(V) = npq$

S.D $(\sigma) = \sqrt{npq}$

$(n-2)-(x-2)$
q
$+np$

Problems

① Find the binomial distribution which has mean 2 and variance $4/3$

np

Soln: W.K.T Mean $(\mu) = np$

Given $\mu = np = 2$

$\therefore np = 2$

Variance $(V) = npq = 4/3$

$npq = 4/3$

$2q = 4/3$

$q = 2/3$

$p+q=1 \Rightarrow p = 1-q = 1-2/3 = \frac{3-2}{3} = \frac{1}{3}$

$\therefore p = 1/3, \quad q = 2/3$

$np = 2$

$n = 2/p = 2/\tfrac{1}{3}.$

$\boxed{n = 6}$

$\therefore$ Binomial probability distribution

$p(x) = n_{C_x}\, p^x\, q^{n-x}$

$p(x) = 6_{C_x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

The distribution of probability is follows

x	0	1	2	3	4	5	6
$p(x)$	$\left(\frac{2}{3}\right)^6$	$6_{C_1}(1/3)(2/3)^5$	$6_{C_2}(1/3)^2(2/3)^4$	$6_{C_3}(1/3)^3(2/3)^3$	$6_{C_4}(1/2)^4(2/3)^2$	$6_{C_5}(1/3)^5(2/3)$	$6_{C_6}(1/2)^6(2/3)^0$

② when a coin is tossed 4 times find the probability of getting
① exactly one head
② atmost 3 heads
③ atleast 2 heads

Sol^n: $p = P(H) = \frac{1}{2}$

w.k.T $p + q = 1$

$q = 1 - p$

$= 1 - 1/2$

$= 1/2$

$\boxed{P = 0.5}$

$n = 4$

w.K.T

$P(x) = {}^nC_x\, p^x q^{n-x}$

$= {}^4C_x (0.5)^x (0.5)^{4-x}$

(i) P(exactly one head) $= P(x=1) = {}^4C_1 (0.5)^1 (0.5)^{4-1}$

$P(x) = {}^nC_x\, p^x q^{n-x}$

$= 4(0.5)(0.5)^3$

$= 0.25$//

(ii) P(atmost 3 heads) $= P(x \leq 3)$

$P(x) = {}^nC_x\, p^x q^{n-x}$

$= P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= {}^4C_0 (0.5)^0 (0.5)^{4-0} + {}^4C_1 (0.5)^1 (0.5)^{4-1}$

$+ {}^4C_2 (0.5)^2 (0.5)^{4-2} + {}^4C_3 (0.5)^3 (0.5)^{4-3}$

$= (1)(1)(0.5)^4 + 4(0.5)(0.5)^3 + 6(0.5)^2(0.5)^2 + 4(0.5)^3(0.5)$

$= 0.0625 + 0.25 + 0.375 + 0.25$

$= 0.9375$//

(iii) P(atleast 2 heads) $= P(x \geq 2)$

$= P(x=2) + P(x=3) + P(x=4)$

$P(x) = {}^nC_x\, p^x q^{n-x}$

$= {}^4C_2 (0.5)^2 (0.5)^{4-2} + {}^4C_3 (0.5)^3 (0.5)^{4-3} + {}^4C_4 (0.5)^4 (0.5)^{4-4}$

$= 6(0.5)^2(0.5)^2 + 4(0.5)^3(0.5) + (1)(0.5)^4 \underset{\downarrow\,(1)}{(0.5)^0}$

$= 6(0.5)^4 + 4(0.5)^4 + (1)(0.5)^4$

$= 0.375$

$= 0.687$

③ The pro
-ed by
$1/10$. If
what 4
(i) Exa
(ii) Atle
(iii) None

Sol^n:

(i) .
P(ex

(ii) P(a

$= 0.375 + 0.25 + 0.0625$

$= \underline{\underline{0.6875}}$

③ The probability that a pen manufactured by a factory be defective is $1/10$. If 12 such pens are manufactured what is the probability that

(i) Exactly two are defective
(ii) Atleast two are defective
(iii) None of them are defective

Soln: probability of a defective pen is $p = \frac{1}{10} = 0.1$ //

probability of non defective pen is $p + q = 1$

$q = 1 - p$

$q = 1 - 0.1$

$q = 0.9$ //

$n = 12$, we have $P(x) = n_{C_x} p^x q^{n-x}$

$P(x) = 12_{C_x} (0.1)^x (0.9)^{12-x}$

(i)

P(exactly 2 are defective)

$= P(x = 2)$

$= 12_{C_2} (0.1)^2 (0.9)^{12-2}$

$= 66 (0.01) (0.9)^{10}$

$= 0.2301$ //

(ii) P(atleast two are defective) $= P(x \geq 2)$

$= 1 - P(x<2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - [{}^{12}C_0 (0.1)^0 (0.9)^{12-0} + {}^{12}C_1 (0.1)^1 (0.9)^{12-1}]$

$= 1 - [(1)(1)(0.9)^{12} + 12(0.1)(0.9)^{11}]$

$= 1 - 0.659$

$= \underline{0.3409}$

(iii) $P(\text{None of them are defective}) = P(x=0)$

$= {}^{12}C_0 (0.1)^0 (0.9)^{12-0}$

$= (1)(1)(0.9)^{12}$

$= 0.2824$ //

④ In a consigment of electric lamps 5% are defective. If a random sample of 8 lamps are inspected what is the probability that one or more lamps are defective?

Sol^{n}: probability of defective lamp

$p = 5\% = \frac{5}{100} = 0.05$

W.K.T $p+q=1 \Rightarrow q = 1-p \Rightarrow q = 1-0.05 = 0.95$

we have $n=8$

& $p(x) = {}^{n}C_x\, p^x q^{n-x}$

$= {}^{8}C_x (0.05)^x (0.95)^{8-x}$

where x denotes defective lamp

$p(\text{one or more lamps defective})$

$= P(x \geq 1)$

$= 1 - P(x<1)$

$= 1 - P(x=0)$

$= 1 - {}^8C_0 (0.05)^0 (0.95)^{8-0}$

$= 1 - (1)(1)(0.95)^8$

$= 1.06634$

$P(x \geq 1) = 0.33657$ //

(or) You can do like this you will get same answer as above

$P(x \geq 1) = P(x=1) + P(x=2) + P(x=3)$
$+ P(x=4) + P(x=5) + P(x=6) +$
$P(x=7) + P(x=8)$

$= {}^8C_1 (0.05) (0.95)^{8-1} + {}^8C_2 (0.05)^2 (0.95)^{8-2}$
$+ {}^8C_3 (0.05)^3 (0.95)^{8-3} + {}^8C_4 (0.05)^4 (0.95)^{8-4}$
$+ {}^8C_5 (0.05)^5 (0.95)^{8-5} + {}^8C_6 (0.05)^6 (0.95)^{8-6}$
$+ {}^8C_7 (0.05)^7 (0.95)^{8-7} + {}^8C_8 (0.05)^8 (0.95)^{8-8}$

$= 8(0.05)(0.95)^7 + 28(0.05)^2(0.95)^6 +$
$56(0.05)^3(0.95)^5 + 70(0.05)^4(0.95)^4$
$+ 56(0.05)^5(0.95)^3 + 28(0.05)^6(0.95)^2$
$+ 8(0.05)^7(0.95)^1 + (1)(0.05)^8 \underset{(1)}{\underset{\downarrow}{(0.95)^0}}$

$= 0.33657$

⑤ The probability that a person aged 60 years will live upto 70 is 0·65 what is the probability that out of 10 persons aged 60 atleast 7 of them will live upto 70.

Soln: Let x be the number of persons aged 60 years living upto 70 years

For this we have

$$p = 0.65$$

$$p + q = 1 \Rightarrow q = 1 - p = 1 - 0.65$$

$$q = 0.35$$

Now we have $n = 10$

& $p(x) = n_{C_x} p^x q^{n-x}$

we have to find $p(x \geq 7)$

$$\boxed{p(x \geq 7) = p(x = 7) + p(x = 8) + p(x = 9) + p(x = 10)}$$

$$= 10_{C_7} (0.65)^7 (0.35)^{10-7} + 10_{C_8} (0.65)^8 (0.35)^{10-8}$$

$$+ 10_{C_9} (0.65)^9 (0.35)^{10-9} + 10_{C_{10}} (0.65)^{10} (0.35)^{10-10}$$

$$= 10_{C_7} (0.65)^7 (0.35)^3 + 10_{C_8} (0.65)^8 (0.35)^2$$

$$+ 10_{C_9} (0.65)^9 (0.35)^1 + 10_{C_{10}} (0.65)^{10} (0.35)^0$$

$$= 0.25221 + 0.17565 + 0.07249 + 0.01346$$

$$= 0.5138$$

(or) you can do like this also will get same answer

$P(x \geq 7) = 1 - P(x < 7)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)]$

$= 0.5138$ //

⑥ The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 lines are choosen at random, what is the probability that ① no line is busy
② All lines are busy
③ Atleast one line is busy
④ Atmost two lines are busy.

Sol^n: Let x denote the number of telephone lines busy.

we have by data $p = 0.1$

$p + q = 1$

$q = 1 - p = 1 - 0.1 = 0.9$

$q = 0.9$

also $n = 10$

we have $p(x) = n_{C_x} p^x q^{n-x}$

$p(x) = 10_{C_x} (0.1)^x (0.9)^{10-x}$

(i) probability that no line is busy

$= p(x=0)$

$= {}^{10}C_0 (0.1)^0 (0.9)^{10-0}$

$= (1)(1)(0.9)^{10}$

$= 0.3487$ //

(ii) probability that all lines are busy

$= p(x=10)$

$= {}^{10}C_{10} (0.1)^{10} (0.9)^{10-10}$

$= (1)(0.1)^{10} (0.9)^{0}$

$= (1)(0.1)^{10}(1)$

$= (0.1)^{10}$

(iii) probability that atleast one line is busy $= P(x \geq 1)$

$= 1 - P(x < 1)$

$= 1 - [P(x=0)]$

$= 1 - [{}^{10}C_0 (0.1)^0 (0.9)^{10}]$

$= 1 - [(1)(1)(0.9)^{10}]$

$= 1 - 0.34867$

$= 0.65132$ //

(iv) probability that atmost 2 lines are busy $= p(x \leq 2)$

1. $= p(x=0) + p(x=1) + p(x=2)$

$= {}^{10}C_0 (0.1)^0 (0.9)^{10-0} + {}^{10}C_1 (0.1)^1 (0.9)^{10-1}$

$+ {}^{10}C_2 (0.1)^2 (0.9)^{10-2}$

$= (1)(1)(0.9)^{10} + 10(0.1)(0.9)^9 + 45(0.1)^2(0.9)^8$

$= 0.9298$

Do yourself

⑦ In a quiz contest of answering 'Yes' or 'NO' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked? Also find the probability of the same if there are 4 options for a correct answer.

Sol: Let x denotes the correct answer

$p = \frac{1}{2} = 0.5$

$p + q = 1 \Rightarrow q = 1 - p = 1 - 0.5 = 0.5$

$q = 0.5$

$n = 10$

we have $P(x) = {}^nC_x \, p^x q^{n-x}$

$= {}^{10}C_x (0.5)^x (0.5)^{10-x}$

$= {}^{10}C_x (0.5)^x (0.5)^{10} (0.5)^{-x}$

$= {}^{10}C_x (0.5)^{10} (0.5)^{x-x}$

$(0.5)^{x-x} = (0.5)^0$

$= 1$

$P(x) = {}^{10}C_x (0.5)^{10}$

probability of getting atleast 6 answers correctly $= P(x \geq 6)$

$= P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$= {}^{10}C_6 (0.5)^{10} + {}^{10}C_7 (0.5)^{10} + {}^{10}C_8 (0.5)^{10} + {}^{10}C_9 (0.5)^{10} + {}^{10}C_{10} (0.5)^{10}$

$= (0.5)^{10}\left[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$

$= (0.5)^{10}\left[210 + 120 + 45 + 10 + 1\right]$

$P(x \geq 6) = \underline{\underline{0.3769}}$

OR

$P(x \geq 6) = 1 - P(x < 6)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)]$

$= 1 - \left[{}^{10}C_0 (0.5)^{10} + {}^{10}C_1 (0.5)^{10} + {}^{10}C_2 (0.5)^{10} + {}^{10}C_3 (0.5)^{10} + {}^{10}C_4 (0.5)^{10} + {}^{10}C_5 (0.5)^{10}\right]$

$= 1 - (0.5)^{10}\left[{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5\right.$

$= 1 - (0.5)^{10}\left[1 + 10 + 45 + 120 + 210 + 252\right]$

$= \underline{\underline{0.3769}}$

OR

$P(x \geq 6) = P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$

we have $P(x) = {}^{n}C_x \, p^x q^{n-x}$

$= {}^{10}C_6 (0.5)^6 (0.5)^4 + {}^{10}C_7 (0.5)^7 (0.5)^3 + {}^{10}C_8 (0.5)^8 (0.5)^2 + {}^{10}C_9 (0.5)^9 (0.5)^1 + {}^{10}C_{10} (0.5)^{10} (0.5)^0$

$= {}^{10}C_6 (0.5)^{10} + {}^{10}C_7 (0.5)^{10} + {}^{10}C_8 (0.5)^{10} + {}^{10}C_9 (0.5)^{10} + {}^{10}C_{10} (0.5)^{10}$

$= (0.5)^{10} [{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}]$

$= (0.5)^{10}(386) = \underline{\underline{0.3769}}$

⑧ In a sampling large number of parts manufactured by a company, the mean number of defectives in samples of 20 is 2, out of 1000 such samples how many would be expected to contain atleast 3 defective parts.

$\underline{\underline{\text{Sol}^{n}:}}$ Given $\mu = np = 2$ by data $n = 20$

$\therefore np = 2 \Rightarrow 20p = 2 \Rightarrow p = \frac{2}{20} = \frac{1}{10}$

$p = \frac{1}{10} = 0.1$ W.K.T $p + q = 1$

$q = 1 - p = 1 - 0.1 = 0.9$

$q = 0.9$

let x denotes the defective part

$p(x \geq 3)$ = probability of atleast 3 defective parts

$p(x \geq 3) = P(x=3) + P(x=4) + P(x=5) \ldots + P(x=20)$

It is lengthy procedure

(OR)

$P(x \geq 3) = 1 - P(x < 3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

we have $p(x) = {}^{n}C_{x} \, p^{x} q^{n-x}$

$= 1 - [\,{}^{20}C_0 (0.1)^0 (0.9)^{20-0}$

$+ {}^{20}C_1 (0.1)^1 (0.9)^{20-1}$

$+ {}^{20}C_2 (0.1)^2 (0.9)^{20-2}\,]$

$$= 1 - \left[{}^{20}C_0 (0.9)^{20} + {}^{20}C_1 (0.9)^{19} (0.1) + {}^{20}C_2 (0.1)^2 (0.9)^{18} \right]$$

$$= 1 - \left[(1)(0.9)^{20} + 20 (0.1)(0.9)^{19} + 190 (0.1)^2 (0.9)^{18} \right]$$

$p(x \geq 3) = 0.323$

Thus Number of defectives in 1000 Sample is 323//

⑨ In 800 families with 5 children each how many family would be expected to have (i) 3 boys
(ii) 5 girls
(iii) either 2 or 3 boys
(iv) atmost 2 girls
by assuming the probabilities for boys and girls to be equal

Sol^n: p = probability of having a boy = $\frac{1}{2}$ = 0.5
q = probability of having a girl = $\frac{1}{2}$ = 0.5
let x denotes the no. of boys ~~in the family~~

$n = 5$

$P(x) = {}^{n}C_x \, p^x q^{n-x}$

$P(x) = {}^{5}C_x (0.5)^x (0.5)^{n-x}$

(i) $P(x=3) = {}^{5}C_3 (0.5)^3 (0.5)^{5-3}$

$= 10 (0.5)^3 (0.5)^2 = 10 (0.5)^5$

$= 0.3125$//

Thy expected number of familiy with 3 boys is $800 \times 0.3125 = 250$ //

(ii) $P(x=5) = 5C_5 (0.5)^5 (0.5)^{5-5}$

$= (1)(0.5)^5 (0.5)^0$

$= (1)(0.03125)(1)$

$= 0.03125$ //

Thy expected no/. of familiy with 5 girls is $800 \times 0.03125 = 25$ //

(iii) $P(x=2) + P(x=3) = 5C_2 (0.5)^2 (0.5)^{5-2}$

$+ 5C_3 (0.5)^3 (0.5)^{5-3}$

$= (10)(0.5)^2 (0.5)^3 + 10(0.5)^3 (0.5)^2$

$= (10)(0.5)^5 + 10(0.5)^5$

=0.5

$= 0.625$ //

=0.5

Thy expected no/. of familiy with 2 or 3 boys is 500 //

(iv) Atmost 2 girls means

(P:$x \leq 2$)

→ family can have 5 boys & 0 girls
or
4 boys & 1 girl.
3 boys & 2 girl

$P(x=5) + P(x=4) + P(x=3)$

no/. of boys

$= 0.03125 + 5C_4 (0.5)^4 (0.5) + 0.3125$

$= 0 \cdot 5$

Expected no. of families with atmost 2 girls is $800 \times 0.5 = 400$

⑩ four coins are tossed 100 times and the following results were obtained fit a binomial distribution for the data & Caluclate the theoretical frequencies

No. of heads	0	1	2	3	4
frequency	5	29	36	25	5

Solⁿ: let x denote the number of heads and f the corresponding frequency since the data is in the form of a frequency distribution we shall first Caluclate the mean.

$$\text{mean}(\mu) = \frac{\sum fx}{\sum f} = \frac{0+29+72+75+20}{100}$$

$$= \frac{196}{100}$$

$$\mu = 1 \cdot 96$$

$\mu = np$ for binomial distribution

four coins $n=4$, $4p = 1 \cdot 96 \Rightarrow p = 0 \cdot 49$

w.k.t $p+q=1$

$q = 1-p$

$q = 1-0.49$

$q = 0 \cdot 51$

we have $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$p(x) = {}^{4}C_{x} (0 \cdot 49)^{x} (0 \cdot 51)^{4-x}$$

Since 4 coins were tossed 100 times expected (theoretical) frequenciy are obtained from $F(x) = 100\,P(x)$

$$= 100\; {}^{4}C_{x}\,(0.49)^{x}(0.51)^{4-x}$$

where $x = 0, 1, 2, 3, 4$

$$F(0) = 100\; {}^{4}C_{0}\,(0.49)^{0}(0.51)^{4-0} = 6.765 \approx 7$$

$$F(1) = 100\; {}^{4}C_{1}\,(0.49)^{1}(0.51)^{4-1} = 25.999 \approx 26$$

$$F(2) = 100\; {}^{4}C_{2}\,(0.49)^{2}(0.51)^{4-2} = 37.47 \approx 37$$

$$F(3) = 100\; {}^{4}C_{3}\,(0.49)^{3}(0.51)^{4-3} = 24.0004 \approx 24$$

$$F(4) = 100\; {}^{4}C_{4}\,(0.49)^{4}(0.51)^{0} = 5.765 \approx 6$$

∴ required theoretical frequenciy are 7, 26, 37, 24, 6

⑪ The probability of Shooter hitting a target is $1/3$. How many times he should shoot so that the probability of hitting the target atleast once is morethan $3/4$.

Sol^n: let p = Probability of hitting a target $= 1/3$

w.k.t $p + q = 1$

$$q = 1 - p = 1 - 1/3 = 2/3$$

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$P(x) = {}^{n}C_{x}\,(1/3)^{x}(2/3)^{n-x}$$

we have to find n ∋;

$$P(x \geq 1) > 3/4$$

$$1 - P(x < 1) > 3/4$$

$$1 - P(x = 0) > 3/4$$

$$1 - ({}^{n}C_{0}\, p^{0} q^{n}) > 3/4$$

$$1 - (q)^{n} > 3/4$$

$$1 - (2/3)^{n} > 3/4$$

$-(+2/3)^n > \frac{3}{4} - 1$

$-(2/3)^n > -\frac{1}{4}$

$(2/3)^n < 1/4$ $\qquad 0.67 \not< 0.25$

$2/3 = 0.67, \quad 1/4 = 0.25$

we can find n by inspection method

$(2/3)^1 = 0.67, \quad (2/3)^2 = 0.44, \quad (2/3)^3 = 0.3,$

$(2/3)^4 = 0.2$

now $0.2 < 0.25$

$\therefore n = 4$

Poisson Distribution:-

It is regarded as the limiting form of the binomial distribution when n is very large $(n \to \infty)$ & p the probability of success is very small $(q \to 0)$ so that np tends to fixed finite constant say m.

Then $p(x) = \frac{e^{-m} m^x}{x!}$, where $m = np$

To show that P.D is a probability f^n it satisfies the conditions ① $p(x) \geq 0$ ② $\sum p(x) = 1$

Now $\sum p(x) = \sum \frac{e^{-m} m^x}{x!}$

let us take $x = 0, 1, 2, 3 \ldots$

x	0	1	2	3	.....
$p(x)$	e^{-m}	$\frac{e^{-m} m}{1!}$	$\frac{e^{-m} m^2}{2!}$	$\frac{e^{-m} m^3}{3!}$	$+ \ldots$

$= e^{-m}\left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\right)$

we have $p(x) \geq 0$ and

$$\sum p(x) = e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + \dots$$

$$= e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots\right\}$$

$$= e^{-m} \cdot e^{m}$$

$$= e^{-m+m} = e^{0}$$

$\sum p(x) = 1$, both the conditions ① $p(x) \geq 0$ ② $\sum p(x) = 1$ are satisfied, hence $p(x)$ is a probability f^n //

Mean & Standard deviation of poisson distribution

$$\text{Mean}(\mu) = \sum_{x=0}^{\infty} x\,p(x)$$

$$= \sum_{x=0}^{\infty} [x] \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x} m^x e^{-m}}{\cancel{x}(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{m^{x-1+1}}{(x-1)!} e^{-m}$$

$$= \sum_{x=1}^{\infty} \frac{m^{x-1} . m . e^{-m}}{(x-1)!}$$

$$= me^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m}\left[1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\dots\right]$$

$$= me^{-m}e^{m}$$

$$= me^{-m+m}$$

$$= me^{0}$$

$$= m$$

∴ mean $(\mu) = m$//

Variance $(V) = \sum_{x=0}^{\infty} x^2 p(x) - \mu^2$ — ①

Now $\sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty}\left[x(x-1)+x\right]p(x)$

$$= \sum_{x=0}^{\infty}\left[x(x-1)p(x) + xp(x)\right]$$

$$= \sum_{x=0}^{\infty} x(x-1)p(x) + \sum_{x=0}^{\infty} xp(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)p(x) + m$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{m^x e^{-m}}{x!} + m$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x(x-1)}\, m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + m$$

$$= \sum_{x=0}^{\infty} \frac{m^x e^{-m}}{(x-2)!} + m$$

$$= \sum_{x=2}^{\infty} \frac{m^{x-2+2} e^{-m}}{(x-2)!} + m$$

$$= \sum_{x=2}^{\infty} \frac{m^{x-2} \cdot m^{2} \cdot e^{-m}}{(x-2)!} + m$$

$$= m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right] + m$$

$$= m^2 e^{-m} e^{m} + m$$

$$= m^2 e^{-m+m} + m$$

$$= m^2 e^{0} + m$$

Variance $= m^2 + m$

$$\therefore \sum x^2 p(x) = m^2 + m$$

Now ① become

Variance $V = m^2 + m - \mu^2$

$\boxed{\text{w.k.t} \quad \mu = m}$

$$= \cancel{m^2} + m - \cancel{m^2}$$

variance, $V = m$

$$S.D(\sigma) = \sqrt{V} = \sqrt{m}$$

hence mean $\mu = m$, Variance$(V) = m$,

& $S.D(\sigma) = \sqrt{m}$

we can say that mean & variance are equall for the poisson distribution

problems

① 2% of the fuses manufactured by a firm are found to be defective find the probability that a box containing 200 fuses contains
(i) no defective fuses
(ii) 3 or more defective fuses.

Sol: p = probability of defective fuse
$= 2/100$
$= 0.02$

$\therefore$ mean no. of defectives $\mu = m = np$
$= 200 \times 0.02$
$\mu = m = 4$
$\therefore m = 4$ //

poisson distribution is given by

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$= \frac{4^x e^{-4}}{x!}$$

$$p(x) = \frac{4^x (0.0183)}{x!}$$

(i) prob(no defective fuse) $= P(x=0)$ (or) $P(0)$

$$= \frac{4^0 (0.0183)}{0!}$$

$$= \frac{(1)0.0183}{(1)}$$

$= 0.0183$ //

(ii) prob.(3 or more defective fuses)
$= P(x \geq 3)$
$= 1 - P(x \lneq 3)$

$$= 1 - [p(0) + p(1) + p(2)]$$

$$= 1 - \left[0.0183 \cdot \frac{4^0}{0!} + 0.0183 \cdot \frac{4^1}{1!} + 0.0183 \cdot \frac{4^2}{2!}\right]$$

$$= 1 - 0.0183\left[\frac{(1)}{(1)} + \frac{4}{1} + \frac{16}{2}\right]$$

$$= 1 - 0.0183\,[1 + 4 + 8]$$

$$= 1 - 0.2379$$

$$= 0.7621$$

② Fit a poisson distribution for the following data & Caluclate the theoretical frequencies.

x	0	1	2	3	4	
f	122	60	15	2	1	

Solⁿ:

$$p(x) = \frac{e^{-m} m^x}{x!}$$

$$\mu = m = \frac{\Sigma fx}{\Sigma f} = \frac{0(122) + (1)(60) + 2(15) + 3(2) + 4(1)}{122 + 60 + 15 + 2 + 1}$$

$$= \frac{60 + 30 + 6 + 4}{200}$$

$$\mu = m = \frac{100}{200}$$

$$\boxed{m = 0.5}$$

Now $p(x) = \frac{e^{-0.5} (0.5)^x}{x!}$

Now we have to find theoretical frequency

Let $f(x) = P(x)\Sigma f$

$$f(x) = P(x)200 = \frac{e^{-0.5}(0.5)^x}{x!} . 200$$

put $x = 0, 1, 2, 3, 4$

$$f(0) = 200\frac{e^{-0.5}(0.5)^0}{0!} = 200 \times 0.6065 = 121.3$$
$$\approx 121$$

$$f(1) = 200\frac{e^{-0.5}(0.5)^1}{1!} = 200 \times 0.6065 \times 0.5$$

$$f(1) = 60.65 \approx 61$$

$$f(2) = 200\frac{e^{-0.5}(0.5)^2}{2!} = \frac{200 \times 0.6065 \times 0.25}{2} = 15.1625$$

$$f(3) = 200\frac{e^{-0.5}(0.5)^3}{3!} = \frac{200 \times 0.6065 \times 0.125}{6}$$

$$f(3) = 2.527 \approx 3$$

$$f(4) = 200\frac{e^{-0.5}(0.5)^4}{4!} = \frac{200 \times 0.6065 \times 0.0625}{24}$$

$$f(4) = 0.3157 \approx 0$$

∴ Theoretical frequencies are 121, 61, 15, 3, 0

③ A no. of accidents in a year to taxi drivers in a city follows a poisson distribution with mean 3. out of 1000 taxi drivers find approximetly the number of the drivers with ① no accident in a year ② morethan 3 accident in a year

By data mean $(\mu) = 3$ i.e $\mu = m = 3$

$\therefore \boxed{m=3}$

poisson distribution is given by

$$p(x) = \frac{m^x e^{-m}}{x!}$$

$$p(x) = \frac{3^x e^{-3}}{x!}$$

(i) no accident in a year

$$P(x=0) = \frac{3^0 e^{-3}}{0!} = \frac{(1)(0.04978)}{1} = 0.04978$$

Number of drivers out of 1000 with no accident in a year is 1000×0.04978

$= 49.78$

$\simeq 50$//

(ii) morethan 3 accidents in a year

$$P(x>3) = 1 - P(x \leq 3)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!}\right]$$

$$= 1 - e^{-3}\left[1 + 3 + \frac{9}{2} + \frac{27}{6}\right]$$

$= 1 - (0.04978)(13)$

$= 1 - (0.64714)$

$= 0.3528$

No. of drivers out of 1000 with morethan 3 accidents in a year

$= 1000 \times 0.3528$

$= 352.86 \simeq 353$//

Do yourself

④ The number of accident per day (x) as recorded in a textile industry over a period of 400 days is given fit a poisson distribution for the data & Caludate the theoretical frequencies

x	0	1	2	3	4	5
f	173	168	37	18	3	1

Solⁿ: Refer problem no. ②

⑤ A Shop has 4 disel generator sets which its hires every day. The demand for a generator set on an average is a poisson variate with value 5/2. Obtain the probability that on a particular day ① there was no demand ② a demand had to be refused

Sol^n: By data mean $\mu = m = 5/2 = 2.5$

$\therefore m = 2.5$

Poisson distribution is given by

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$P(x) = \frac{(2.5)^x e^{-2.5}}{x!}$$

(i) No demand for generator

$$P(x=0) = P(0) = \frac{(2.5)^0 e^{-2.5}}{0!} = 0.082085$$

(ii) If a demand had to be refused, there should have been a demand for morethan 4 generators.

we have to find $P(x>4)$

$$P(x>4) = 1 - P(x \le 4)$$

$$= 1 - \left[P(0) + P(1) + P(2) + P(3) + P(4) \right]$$

$$= 1 - \left[\frac{(2.5)^0 e^{-2.5}}{0!} + \frac{(2.5)^1 (e^{-2.5})}{1!} + \frac{(2.5) e^{-2.5}}{2!} + \frac{(2.5)^3 e^{-2.5}}{3!} + \frac{(2.5)^4 e^{-2.5}}{4!} \right]$$

$$= 1 - e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$$

$P(x>4) = 0.10882$

⑥ The probability that a news reader commits no mistake in reading the news is $1/e^3$. Find the probability that on a particular news broadcast he commits ① only 2 mistakes

② morethan 3 mistakes

③ atmost 3 mistakes

Soln: By data

probability that a news reader commits no mistake $P(x=0)$ or $P(0) = 1/e^3 = e^{-3}$

Now poisson distribution is

$$P(x) = \frac{m^x e^{-m}}{x!}$$

where x denotes commiting mistake

we have $e^{-m} = e^{-3} \Rightarrow -m = -3 \Rightarrow m = 3$

$$\therefore P(x) = \frac{3^x e^{-3}}{x!}$$

① probability of commiting 2 mistakes

$$P(x=2) \text{ or } P(2) = \frac{3^2 e^{-3}}{2!} = 0.0498 \times \frac{9}{2}$$

$P(2) = 0.2241$

② probability of committing morethan 3 mistakes is $P(x>3)$

$$P(x>3) = 1 - P(x \leq 3)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$= 1 - e^{-3}\left[1 + 3 + 9/2 + 27/6\right]$

$= 1 - e^{-3}(13)$

$p(x>3) = 0.3528$ //

③ probability of commiting atmost 3 mistakes

$P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

$= e^{-3} + \frac{e^{-3} 3^1}{1!} + \frac{e^{-3} 3^2}{2!} + \frac{e^{-3} 3^3}{3!}$

$= e^{-3}\left[1 + 3 + \frac{9}{2} + \frac{27}{6}\right]$

$p(x \leq 3) = 0.6472$

⑦ If the probability of a bad reaction from a Certain injection is 0.001, determine the chance that out of 2000 individuals morethan two will get a bad reaction.

Sol^n: poisson distribution is given by

$p(x) = \frac{m^x e^{-m}}{x!}$

$n = 2000$

$p = 0.001$

mean $\mu = m = np = 2000 \times 0.001$

$m = 2$

we have to find $P(x > 2)$

$P(x>2) = 1 - P(x \leq 2)$

$= 1 - [P(0) + P(1) + P(2)]$

w.K.T $p(x) = \frac{2^x e^{-2}}{x!}$

$= 1 - \left[\frac{e^{-2} 2^{0}}{0!} + \frac{e^{-2} 2^{1}}{1!} + \frac{e^{-2} 2^{2}}{2!} \right]$

$= 1 - e^{-2} \left[1 + 2 + 4/2 \right]$

$= 1 - e^{-2} \left[1 + 2 + 2 \right]$

$p(x>2) = 0.3233$ //

⑧ Do yourself,

In a certain factory turning out razor blady there is a small probability of 1/500 for any blade to be defective the blady are supplied in packets of 10. Use poisson distribution to caluclate the approximate no. of packets containing (i) no defective

(ii) one defective

(iii) two defective blady in a consignment of 10,000 packets.

Soln. probability of a defective blade

$p = 1/500 = 0.002$

$n = 10$

Mean $\mu = m = np = 10 \times 0.002 = 0.02$

$m = 0.02$

$p(x) = \frac{m^x e^{-m}}{x!}$

$p(x) = \frac{(0.02)^x e^{-0.02}}{x!}$ | $e^{-0.02} = 0.9802$

$p(x) = \frac{(0.02)^x (0.9802)}{x!}$

① prob. of no defective $= P(x=0) = \frac{0.9802 \times (0}{0!}$

$p(x=0)$ or $p(0) = 0.9802$

No/. of packets containing no defective blades out of 10,000

$= 0.9802 \times 10,000$

$= 9,802$//

② probability of one defective $= P(x=1)$

(or) $P(1) = \frac{0.9802 \times (0.02)^1}{1!} = 0.0196$

No/. of packets containing one defective blade out of $10,000 = 0.0196 \times 10,000$

$= 196$

③ probability of two defective $= P(x=2)$ or $P(2)$

$= \frac{0.9802 \times (0.02)^2}{2!}$

$P(2) = 0.00019$

No/. of packets containing two defective blades out of $10,000 = 10,000 \times 0.00019$

$= 1.9$

≈ 2//

OR

you can solve like this

$p = 1/500 = 0.002$

mean $m = np = 10 \times 0.002 = 0.02$//

$P(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.02)^x e^{-0.02}}{x!}$

Let $f(x) = 10,000\, P(x)$

$f(x) = 10,000 \frac{(0.02)^x e^{-0.02}}{x!}$

① probability of no defective $f(0) =$

$= 10{,}000 \frac{(0.02)^0 e^{-0.02}}{0!} = 9802$//

(ii) probability of one defective $= f(1)$.

$= 10{,}000 \times \frac{(0.02) e^{-0.02}}{1!}$

$= 196$

(iii) probability of two defectives

$f(2) = 10{,}000 \frac{(0.02)^2 e^{-0.02}}{2!} = 1.96 \approx 2$//

(9) If X follows a poisson variate ∋: $P(X=2) = 2/3\, P(X=1)$, find $p(x=0)$ and $p(x=3)$

Soln: we have $P(X=x) = \frac{m^x e^{-m}}{x!}$

By data

$P(X=2) = 2/3\, P(X=1)$

$\frac{\cancel{m^2} \cancel{e^{-m}}}{2!} = \frac{2}{3} \frac{\cancel{m} \cancel{e^{-m}}}{1!}$

$\frac{m}{2} = \frac{2}{3} \Rightarrow m = 4/3$//

hence $p(x=x) = \frac{m^x e^{-m}}{x!}$, $m = 4/3$

$P(X=x) = \frac{(4/3)^x e^{-4/3}}{x!}$

$P(X=0) = \frac{(4/3)^0 e^{-4/3}}{0!} = e^{-4/3} = 0.2636$

$P(X=3) = \frac{(4/3)^3 e^{-4/3}}{3!} = 0.10414$

⑩ If x is a poisson variate $\ni$; $p(x=2) = 9\,p(x=4) + 90\,p(x=6)$, compute the mean & variance of poisson distribution.

Sol^n: we have $p(x) = \dfrac{m^x e^{-m}}{x!}$

by using the data we have

$$p(x=2) = 9\,p(x=4) + 90\,p(x=6)$$

$$\frac{m^2 e^{-m}}{2!} = 9\,\frac{m^4 e^{-m}}{4!} + 90\,\frac{m^6 e^{-m}}{6!}$$

$$\frac{\cancel{m^2}\cancel{e^{-m}}}{2} = \cancel{m^2}\,\cancel{e^{-m}}\left[\frac{9m^2}{24} + \frac{90\,m^4}{720}\right]$$

$$\frac{1}{2} = \frac{\overset{3}{\cancel{9}}\,m^2}{\underset{8}{\cancel{24}}} + \frac{m^4}{8}$$

$$\frac{1}{2} = \frac{3}{8}m^2 + \frac{m^4}{8}$$

$$1 = \frac{3}{4}m^2 + \frac{m^4}{4}$$

$$4 = 3m^2 + m^4$$

$$m^4 + 3m^2 - 4 = 0$$

$-4m^4$

$+4m^2 \quad m^2$

[illegible]

$$m^4 + 4m^2 - m^2 - 4 = 0$$

$$m^2(m^2+4) - 1(m^2+4) = 0$$

$$(m^2+4)(m^2-1) = 0$$

$$m^2+4 = 0 \quad \text{or} \quad m^2 - 1 = 0$$

$$m^2 = -4, \qquad m^2 = 1$$

$$m = \pm 2i \qquad m = \pm 1$$

Neglect complex root & (value) negative values.

$\therefore m = 1$

variance and mean are equal in poisson distribution

$\therefore m = \text{variance} = 1$

variance and mean are equal in poisson distribution

$$\therefore m = \text{variance} = 1$$

Contineous probability distribution

If every x belonging to the range of contineous random variable 'x' we assign a real no, $f(x)$ satisfying the conditions ① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then $f(x)$ is called a contineous probability function or probability density function

Cumulative distribution function:-

If X is a contineous random variable with probability density function $f(x)$ then $F(x)$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$ is

called cumulative distribution function

NOTE: $\frac{d}{dx}[F(x)] = f(x)$

$$P(x \geq r) = \int_{r}^{\infty} f(x)\,dx$$

$$P(x < r) = 1 - P(x \geq r)$$

$$P(x < r) = 1 - \int_{r}^{\infty} f(x)\,dx$$

Mean & Variance of Contineous Probability distribution

$$\text{mean } (\mu) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

Exponential distribution:

The contineous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x>0 \\ 0 & \text{otherwise, where } \alpha>0 \end{cases}$$

is known as exponential distribution

If $f(x)>0$ & we have

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\, dx$$

$$= \alpha \int_{0}^{\infty} e^{-\alpha x}\, dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^{\infty}$$

$$= \frac{\alpha}{-\alpha} \left[e^{-\alpha x} \right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0} \right]$$

$= -[0 - 1]$

$= 1$

$$\therefore \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

f(x) satisfy both the conditions required for a Contineous probability function / probability density function.

mean & Variance of Exponential distribution ☆

$$\text{mean}(\mu) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{+0}^{\infty} x \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_{+0}^{\infty} x e^{-\alpha x}\,dx$$

Applying Bernoulli's rule of integration by parts we have

$$\mu = \alpha \left[x \left(\frac{e^{-\alpha x}}{-\alpha} \right) - (1) \left(\frac{e^{-\alpha x}}{\alpha^2} \right) \right]_0^{\infty}$$

$$\mu = \alpha \left[(0 - 0) - \left(0 - \left(\frac{+1}{\alpha^2} \right) \right) \right]$$

$$= \alpha \left[0 + \frac{1}{\alpha^2} \right]$$

$$\mu = \alpha\left(\tfrac{1}{\alpha^2}\right)$$

$$\boxed{\mu = \tfrac{1}{\alpha}}$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

$$\sigma^2 = \int_0^{\infty} (x-\mu)^2 \alpha e^{-\alpha x}\, dx$$

$$\sigma^2 = \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x}\, dx$$

Applying Bernoulli's rule we have

$$\sigma^2 = \alpha\left[(x-\mu)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)\frac{e^{-\alpha x}}{\alpha^2} + 2\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^{\infty}$$

$$= \alpha\left[-\frac{1}{\alpha}\left(0-\mu^2\right) - \frac{2}{\alpha^2}\left(0-(-\mu)\right) - \frac{2}{\alpha^3}\left(0-1\right)\right]$$

$$= \alpha\left[\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right]$$

But $\mu = \frac{1}{\alpha}$

$$= \alpha\left[\frac{1/\alpha^2}{\alpha} - \frac{2\cdot 1/\alpha}{\alpha^2} + \frac{2}{\alpha^3}\right]$$

$$= \alpha\left[\frac{1}{\alpha^3} - \cancel{\frac{2}{\alpha^3}} + \cancel{\frac{2}{\alpha^3}}\right]$$

$$\sigma^2 = \frac{1}{\alpha^2}$$

$\sigma = \frac{1}{\alpha}$

Thus for exponential distribution
mean $(\mu) = \frac{1}{\alpha}$; S.D $(\sigma) = \frac{1}{\alpha}$, variance $\sigma^2 = 1/\alpha^2$

NOTE: mean = S.D for the exponential distribution.

Problems

① Find which of the following functions is a probability density function.

(i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(iii) $f_3(x) = \begin{cases} |x|, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(iv) $f_4(x) = \begin{cases} 2x & 0 < x \leq 1 \\ 4-4x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Sol: To show that given functions are p.d.f it should satisfiy the conditions

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

① clearly $f(x) \geq 0$

$\int_{-\infty}^{\infty} f_1(x)\,dx = \int_0^1 f_1(x)\,dx = \int_0^1 2x\,dx = \left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_0^1$

$= [x^2]_0^1$

$= [1-0]$

$\int_{-\infty}^{\infty} f_1(x)\,dx = 1$

$\therefore f_1(x)$ is a p.d.f

(ii) The given function can be written in the form

$$f_2(x) = \begin{cases} 2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In $-1 < x < 0$, $f_2(x) = 2x$ is lessthan zero

$$\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-1}^{1} f_2(x)\,dx$$

$$= \int_{-1}^{1} 2x\,dx$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_{-1}^{1}$$

$$= [1-(-1)^2]$$

$$= 1-1$$

$$= \underline{\underline{0}}$$

both the conditions are not satisfied.

$\therefore f_2(x)$ is not a p.d.f.

(iii) Evidently $f_3(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{1} f_3(x)\,dx = \int_{-1}^{1} |x|\,dx$$

$$|x| = \begin{cases} -x & -1 < x < 0 \\ x & 0 < x < 1 . \end{cases}$$

$$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$$

$$= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$= -\frac{1}{2}\left[0-(-1)^2\right] + \frac{1}{2}\left[1-0\right]$$

$$= -\frac{1}{2}\left[-1\right] + \frac{1}{2}\left[1\right]$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$ //

$\therefore$ $f_3(x)$ is p.d.f

(iv) $f_4(x) = 2x > 0$ in $0 < x \leq 1$

$f_4(x) = 4 - 4x$ is negative in $1 < x < 2$

The first condition is not satisfied

$f_4(x)$ is not a p.d.f

② Find the Value of c such that

$$f(x) = \begin{cases} \frac{x}{6}+c, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

is a probability density function
also find $P(1 \le x \le 2)$

p.d.f is valid

Sol^n: ① $f(x) \geq 0$ if $c \geq 0$

also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 \left(\frac{x}{6}+c\right)dx = 1$$

$$\left[\frac{x^2}{6\times 2}+cx\right]_0^3 = 1$$

$$\left(\frac{3^2}{12}+c\cdot 3\right)-0 = 1$$

$$\frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{12}}}+3c = 1$$

$$\frac{3}{4}+3c = 1$$

$$3c = 1-\frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$$

$$3c = \frac{1}{4} \Rightarrow c = \frac{1}{12}$$

Now $P(1 \le x \le 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 \left(\frac{x}{6}+c\right)dx$$

$$= \int_1^2 \left(\frac{x}{6}+\frac{1}{12}\right)dx$$

$$= \left[\frac{x^2}{12} + \frac{1}{12}\cdot x\right]_1^2$$

$$= \left(\frac{4}{12} + \frac{2}{12}\right) - \left(\frac{1}{12} + \frac{1}{12}\right)$$

$$= \left(\frac{1}{3} + \frac{1}{6}\right) - \left(\frac{2}{12}\right)$$

$$= \frac{2+1}{6} - \frac{\cancel{2}^{1}}{\cancel{12}_{6}}$$

$$= \frac{3}{6} - \frac{1}{6}$$

$$= \frac{2}{6}$$

$P(1 \le x \le 2) = \frac{1}{3}$ //

OR $= \frac{1}{12}\left[x^2 + x\right]_1^2$

$$= \frac{1}{12}\left[(2^2+2) - (1+1)\right]$$

$$= \frac{1}{12}\left[6-2\right]$$

$$= \frac{1}{\cancel{12}_{3}} \times \cancel{4}^{1}$$

$P(1 \le x \le 2) = 1/3$

③ Find the Constant K Such that $f(x) = \begin{cases} Kx^2 ; & 0 < x < 3 \\ 0 , & \text{otherwise} \end{cases}$ is a probabality density function also compute

① $P(1 < x < 2)$ ② $P(x \le 1)$ ③ $P(x > 1)$

④ mean ⑤ variance

Solⁿ: probability density function is valid $f(x) \ge 0$ if $K \ge 0$.

also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{3} Kx^2\,dx = 1$$

$$K\int_0^3 x^2\,dx = 1$$

$$K\left[x^3/3\right]_0^3 = 1$$

$K\left[\frac{\cancel{3}^3}{\cancel{3}} - 0\right] = 1$

$K\left[3^2\right] = 1$

$9K = 1$

$K = \frac{1}{9}$

(i) $p(1 < x < 2) = \int_1^2 f(x)\,dx$

$= \int_1^2 Kx^2\,dx = \int_1^2 \frac{1}{9}x^2\,dx$

$= \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2$

$= \frac{1}{9}\left[\frac{2^3}{3} - \frac{1}{3}\right]$

$= \frac{1}{9}\left[\frac{8}{3} - \frac{1}{3}\right] = \frac{1}{9} \times \frac{7}{3}$

$= \frac{7}{27}$ //

(ii) $p(x \leq 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^1$

$= \left[\frac{1}{27} - 0\right]$

$P(x \leq 1) = \frac{1}{27}$ //

(iii) $P(x > 1) = \int_1^3 f(x)\,dx = \int_1^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^3$

$= \left[\frac{3^3}{27} - \frac{1}{27}\right]$

$= \left[\frac{27}{27} - \frac{1}{27}\right]$

$= \left[1 - 1/27\right]$

$p(x>1) = \frac{26}{27}$ //

(iv) mean $= \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \int_0^3 x \cdot \frac{x^2}{9}\, dx$

$= \int_0^3 \frac{x^3}{9}\, dx$

$= \frac{1}{9}\int_0^3 x^3\, dx = \frac{1}{9}\left[\frac{x^4}{4}\right]_0^3$

$= \frac{1}{36}\left[3^4 - 0\right]$

$= \frac{1}{36}(81)$

$\mu = \frac{9}{4}$ //

(v) Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (\mu)^2$

$= \int_0^3 x^2 \cdot \frac{x^2}{9}\, dx - \left(\frac{9}{4}\right)^2$

$= \int_0^3 \frac{x^4}{9}\, dx - \frac{81}{16}$

$$= \left[\frac{x^5}{9\times5}\right]_0^3 - \frac{81}{16}$$

$$= \left(\frac{3^5}{45} - 0\right) - \frac{81}{16}$$

$$= \frac{\cancel{243}^{\,81}}{\cancel{45}_{\,15}} - \frac{81}{16}$$

$$= \frac{81}{15} - \frac{81}{16} = \frac{81}{240} = \frac{27}{80}$$

$$V = \frac{27}{80} \,//$$

④ A random variable x has the following density function

$$p(x) = \begin{cases} kx^2, & -3 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Evaluate k & find ① $p(1 \le x \le 2)$

② $p(x \le 2)$

③ $p(x > 1)$

Sol^n: we must have ① $p(x) \geq 0$ if $k \geq 0$

② $\int_{-\infty}^{\infty} p(x)\,dx = 1$

i.e $\int_{-3}^{3} kx^2\,dx = 1$

$$k\int_{-3}^{3} x^2\,dx = 1$$

$$k\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$$

$$\frac{K}{3}\left[3^3-(-3)^3\right]=1$$

$$\frac{K}{3}\left[27-(-27)\right]=1$$

$$\frac{K}{3}\left[27+27\right]=1$$

$$\frac{K}{\cancel{3}}\left[\overset{18}{\cancel{54}}\right]=1$$

$$18K=1$$

$$K=\frac{1}{18}//$$

① $P(1 \le x \le 2) = \int_1^2 Kx^2\,dx = \int_1^2 \frac{1}{18}x^2\,dx$

$$=\frac{1}{18}\int_1^2 x^2\,dx$$

$$=\frac{1}{18}\left[\frac{x^3}{3}\right]_1^2$$

$$=\frac{1}{18}\left[\frac{2^3}{3}-\frac{1}{3}\right]$$

$$=\frac{1}{18}\left[\frac{8}{3}-\frac{1}{3}\right]$$

$$=\frac{1}{18}\times\frac{7}{3}$$

$$=\frac{7}{54}//$$

② $P(x \le 2) = \int_{-3}^2 Kx^2\,dx = \int_{-3}^2 \frac{1}{18}x^2\,dx = \frac{1}{18}\int_{-3}^2 x^2\,dx$

$$=\frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^2$$

$$=\frac{1}{18}\left[\frac{2^3}{3}-\frac{(-3)^3}{3}\right]$$

$$= \frac{1}{18}\left[\frac{8}{3} - \frac{(-27)}{3}\right]$$

$$= \frac{1}{18}\left[\frac{8}{3} + \frac{27}{3}\right]$$

$$= \frac{1}{18} \times \frac{35}{3}$$

$$= \frac{35}{54} //$$

③ $P(x>1) = \int_1^3 K x^2\,dx = \int_1^3 \frac{1}{18} x^2\,dx$

$$= \frac{1}{18}\int_1^3 x^2\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{1}{18}\left[\frac{3^3}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{18}\left[\frac{27}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{18} \times \frac{26}{3}$$

$$= \frac{26}{54}$$

$$P(x>1) = \frac{13}{27}$$

⑤ Find the CDF (cumulative distribution function) for probability density function of random variable X.

(i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0 & ; \text{otherwise} \end{cases}$

(ii) $$f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Soln: If $f(x)$ is the probability density function then c.d.f $= F(x) = \int_{-\infty}^{x} f(x)\,dx$

(i) $$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} f(x)\,dx$$

$$F(x) = \int_{0}^{x} (6x - 6x^2)\,dx$$

$$= \left[\frac{6x^2}{2} - \frac{6x^3}{3} \right]_0^x$$

$$= \left[3x^2 - 2x^3 \right]_0^x$$

$$F(x) = 3x^2 - 2x^3$$

$\therefore$ c.d.f $= 3x^2 - 2x^3$ if $0 \le x \le 1$ //

(ii) $$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} f(x)\,dx$$

$= \int_0^x \frac{x}{4} e^{-x/2} \, dx$

Apply bernoulli's rule

$= \frac{1}{4} \int_0^x x \, e^{-x/2} \, dx$

$= \frac{1}{4} \left[x \frac{e^{-x/2}}{-1/2} - (1) \frac{e^{-x/2}}{-\frac{1}{2} \times -1/2} \right]_0^x$

$= \frac{1}{4} \left[-2xe^{-x/2} - 4e^{-x/2} \right]_0^x$

$= \frac{1}{4} \left[\left\{ -2xe^{-x/2} - 4e^{-x/2} \right\} - \left\{ 0 - 4e^{0} \right\} \right]$

$= \frac{1}{4} \left[-2(xe^{-x/2} + 2e^{-x/2}) + 4 \right]$

$= \frac{1}{4} \left[-2e^{-x/2} (x+2) + 4 \right]$

$= -\frac{1}{2} e^{-x/2} (x+2) + 1$

$= -\frac{1}{2} e^{-x/2} . x + e^{-x/2} + 1$

$F(x) = 1 - \frac{x}{2} e^{-x/2} - e^{-x/2}$ if $0 < x < \infty$

⑥ A Contineous random variable has the distribution function

$$F(x) = \begin{cases} 0, & x \leq 1 \\ c(x-1)^4, & 1 \leq x \leq 3 \\ 1, & x > 3 \end{cases}$$

Find c and also the p.d.f

Sol^n W.K.T p.d.f $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 0 \ ; & x \leq 1 \\ 4c(x-1)^3 \ ; & 1 \leq x \leq 3 \\ 0 \ , & x > 3 \end{cases}$$

① $f(x) \geq 0$ for $c \geq 0$

② we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_1^3 f(x)\,dx = 1, \quad \int_1^3 4c(x-1)^3\,dx = 1$$

$$4c\int_1^3 (x-1)^3\,dx = 1$$

$$4c\left[\frac{(x-1)^4}{\cancel{4}}\right]_1^3 = 1$$

$$c\left[(x-1)^4\right]_1^3 = 1$$

$$c\left[(3-1)^4 - (1-1)^4\right] = 1$$

$$c\left[2^4 - 0\right] = 1$$

$$c\,[16] = 1$$

$$c = 1/16 \,//$$

Thus p.d.f $f(x) = (x-1)^3 \cdot 4c = (x-1)^3 \cdot 4 \cdot \frac{1}{\cancel{16}_{4}}$

$$= \frac{(x-1)^3}{4} \text{ where } 1 \leq x \leq 3$$

⑦ Find K so that the following function Can serve as a p.d.f of a random variable $f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ Kx\,e^{-4x^2} & \text{for } x > 0 \end{cases}$

Sol^n: we must have $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{0}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} Kx\,e^{-4x^2}\,dx = 1$$

$$\Rightarrow K\int_{0}^{\infty} x\,e^{-4x^2}\,dx = 1$$

put $4x^2 = t$

D.w.r. to x

$8x\,dx = dt$

$x\,dx = \frac{1}{8}\,dt$

when x varies from 0 to ∞

t also varies from 0 to ∞

$$\Rightarrow K\int_{0}^{\infty} e^{-t}\,\frac{dt}{8} = 1$$

$$\Rightarrow \frac{K}{8}\left[\frac{e^{-t}}{-1}\right]_{0}^{\infty} = 1$$

$$\Rightarrow \frac{-K}{8}\left[e^{-t}\right]_{0}^{\infty} = 1$$

$$\Rightarrow -\frac{K}{8}\left[0-1\right]=1$$

$$\frac{K}{8}=1$$

$$\underline{\underline{K=8}}$$

⑧ The kilometre run (in thousands of kms) without any sort of problem in respect of a Certain vehicle is a random variable having p.d.f

$$f(x)=\begin{cases}\frac{1}{40}e^{-x/40}, & x\geq 0\\ 0, & x\leq 0\end{cases}$$

find the probabality that vehicle is trouble free

① atleast for 25000 kms

② atmost for 25000 kms

③ between 16000 to 32000 kms

Sol: Here x is the random Variable representing kilometre in multiples of 1000 regarding trouble free run by the vehicle

① To find $P(x\geq 25)$

$$P(x\geq 25)=1-P(x<25)$$

$$P(x\geq 25)=1-\int_0^{25}\frac{1}{40}e^{-x/40}\,dx$$

$$=1-\cancel{\frac{1}{40}}\times\left[\frac{e^{-x/40}}{\cancel{-1/40}}\right]_0^{25}$$

$$=1+\left[e^{-x/40}\right]_0^{25}$$

$$= 1 + \left[e^{-25/40} - e^{0}\right]$$

$$= 1 + \left[e^{-5/8} - 1\right]$$

$$= \underline{\underline{e^{-5/8}}}$$

(ii) $P(x \le 25)$

$$P(x \le 25) = \int_{0}^{25} \frac{1}{40} e^{-x/40}\, dx$$

$$= \frac{1}{\cancel{40}}\left[\frac{e^{-x/40}}{-\cancel{1/40}}\right]_{0}^{25}$$

$$= -\left[e^{-25/40} - e^{0}\right]$$

$$= -\left[e^{-5/8} - 1\right]$$

$$= -e^{-5/8} + 1$$

$P(x \le 25) = 0.4647$ //

(iii) $P(16 \le x \le 32) = \int_{16}^{32} \frac{1}{40} e^{-x/40}\, dx$

$$= \frac{1}{\cancel{40}}\left[\frac{e^{-x/40}}{-\cancel{1/40}}\right]_{16}^{32}$$

$$= -\left[e^{-x/40}\right]_{16}^{32}$$

$$= -\left[e^{-32/40} - e^{-16/40}\right]$$

$$= -\left[e^{-4/5} - e^{-2/5}\right]$$

$$= 0.221 //$$

⑨ If x is an exponential variate with mean 3 find ① $p(x>1)$
② $p(x<3)$

Sol: p.d.f of the exponential distribution is given by $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$

mean of exponential distribution is

$$\mu = \frac{1}{\alpha}$$

given $\mu = 3$

$$\frac{1}{\alpha} = 3 \Rightarrow \alpha = 1/3$$

hence $f(x) = \begin{cases} \frac{1}{3} e^{-\frac{1}{3}x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$

① $P(x>1) = 1-P(x\leq 1)$

$$= 1 - \int_0^1 f(x)\,dx$$

$$= 1 - \int_0^1 \frac{1}{3} e^{-\frac{1x}{3}}\,dx$$

$$= 1 - \cancel{\frac{1}{3}}\left(\frac{e^{-\frac{1}{3}x}}{\cancel{-\frac{1}{3}}}\right)_0^1$$

$$= 1 + \left[e^{-1/3} - e^0\right]$$

$$= \cancel{1} + e^{-1/3} - \cancel{1}$$

$$= e^{-1/3} = 0.7165 //$$

② $P(x<3) = \int_0^3 f(x)\,dx = \int_0^3 \frac{1}{3} e^{-x/3}\,dx$

$$= \cancel{\frac{1}{3}}\left[\frac{e^{-x/3}}{\cancel{-\frac{1}{3}}}\right]_0^3$$

$$= -\left[e^{-1} - e^0\right] = -\left[e^{-1} - 1\right] = 1 - e^{-1}$$

$P(x<3) = 0.6321$

⑩ In a Certain town, the duration of a shower is exponentially distributed with mean 5 minutes. what is the probability that a shower will last for ① 10 minutes or more
② Lessthan 10 minutes
③ between 10 & 12 minutes

Solⁿ: Given $\mu = 5$

$\mu = \frac{1}{\alpha}$, mean of exponential distribution

$\therefore \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5} = 0.2$

probability density function of exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = 0.2 e^{-0.2x}$$

we have to find

① $p(x \geq 10) = \int_{10}^{\infty} f(x)\, dx$

$$= \int_{10}^{\infty} 0.2 e^{-0.2x}\, dx$$

$$= \cancel{0.2}\left[\frac{e^{-0.2x}}{\cancel{-0.2}}\right]_{10}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-0.2 \times 10}\right]$$

$$p(x \geq 10) = -\left[0 - e^{-2}\right] = e^{-2} = \underline{\underline{0.1353}}$$

(ii) $P(x<10) = \int_0^{10} f(x)\,dx$

$$= \int_0^{10} 0.2\, e^{-0.2x}\, dx$$

$$= \cancel{0.2}\left[\frac{e^{-0.2x}}{-\cancel{0.2}}\right]_0^{10}$$

$$= -\left[e^{-0.2\times 10} - e^{0}\right]$$

$$= -\left[e^{-2} - 1\right]$$

$$= 1 - e^{-2}$$

$$= 0.8647 \,//$$

(iii) $P(10<x<12) = \int_{10}^{12} f(x)\,dx$

$$= \int_{10}^{12} 0.2\, e^{-0.2x}\, dx$$

$$= \cancel{0.2}\left[\frac{e^{-0.2x}}{-\cancel{0.2}}\right]_{10}^{12}$$

$$= -\left[e^{-0.2\times 12} - e^{-0.2\times 10}\right]$$

$$= -\left[e^{-2.4} - e^{-2}\right]$$

$$= \underline{\underline{0.0446}}$$

(11) The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. find the probability that a random call made from this booth ① ends less than 5 minutes

② b/w 5 and 10 minutes

Sol: Exponential distribution of mean is $\mu = \frac{1}{\alpha}$

Given $\mu = 5$

$$\frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5} = 0.2$$

p.d.f of exponential distribution is given by $f(x) = \begin{cases} \alpha e^{-\alpha x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

$$f(x) = \alpha e^{-\alpha x}$$

$$f(x) = 0.2 e^{-0.2x}$$

① $P(x<5) = \int_0^5 f(x)\,dx$

$$= \int_0^5 0.2 e^{-0.2x}\,dx$$

$$= 0.2 \int_0^5 e^{-0.2x}\,dx$$

$$= \cancel{0.2}\left[\frac{e^{-0.2x}}{-\cancel{0.2}}\right]_0^5$$

$$= -\left[e^{-0.2\times5} - e^{-0.2\times0}\right]$$

$$= -\left[e^{-1} - e^{0}\right]$$

$$= -\left[e^{-1} - 1\right]$$

$$= 0.6321$$

② $P(5 < x < 10) = \int_5^{10} f(x)\,dx$

$$= \int_5^{10} 0.2\,e^{-0.2x}\,dx$$

$$= 0.2 \int_5^{10} e^{-0.2x}\,dx$$

$$= \cancel{0.2}\left[\frac{e^{-0.2x}}{-\cancel{0.2}}\right]_5^{10}$$

$$= -\left[e^{-0.2\times 10} - e^{-0.2\times 5}\right]$$

$$= -\left[e^{-2} - e^{-1}\right]$$

$$= 0.2325$$

Do yourself

⊛ If x is an exponential variate with mean 5, evaluate

① $P(0 < x < 1)$ Ans : 0.1813

② $P(-\infty < x < 10)$: 0.8647

③ $P(x \le 0 \text{ or } x \ge 1)$: 0.8187

$\hookrightarrow P(x \le 0) + P(x \ge 1)$

$\hookrightarrow 0 + \int_1^{\infty} f(x)\,dx$

⑫ Find K so that $f(x) = \begin{cases} Kxe^{-x}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

is a p.d.f find the mean.

Soln: $f(x) \geq 0$ if $K \geq 0$

we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Kxe^{-x}\,dx = 1$$

$$K\int_0^1 xe^{-x}\,dx = 1$$

apply Bernoulli's rule

$$K\left[x\frac{e^{-x}}{-1} - e^{-x}\right]_0^1 = 1$$

$$K\left[\left\{\frac{e^{-1}}{-1} - e^{-1}\right\} - \left\{0 - e^{0}\right\}\right] = 1$$

$$K\left[-1/e - 1/e + 1\right] = 1$$

$$K\left[-2/e + 1\right] = 1$$

$$K\left[1 - 2/e\right] = 1$$

$$K = \frac{1}{1-2/e} = \frac{e}{e-2}$$

$$K = \frac{e}{e-2}$$ //

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x \cdot \frac{e}{e-2}\, x e^{-x}\, dx$$

$$= \frac{e}{e-2} \int_0^1 x^2 e^{-x}\, dx$$

$$= \frac{e}{e-2} \left[x^2 \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{(-1)(-1)} + 2 \frac{e^{-x}}{(-1)(-1)(-1)} \right]_0^1$$

$$= \frac{e}{e-2} \left[\left\{ \frac{e^{-1}}{-1} - 2\frac{e^{-1}}{1} + 2\frac{e^{-1}}{-1} \right\} - \left\{ 0 - 0 - 2 \right\} \right]$$

$$= \frac{e}{e-2} \left[-e^{-1} - 4e^{-1} + 2 \right]$$

$$= \frac{e}{e-2} \left[2 - 5e^{-1} \right]$$

$$= \frac{e}{e-2} \left[2 - 5/e \right]$$

$$= \frac{\cancel{e}}{e-2} \left[\frac{2e-5}{\cancel{e}} \right]$$

$$= \frac{2e-5}{e-2} \,//$$

(13) Is the following function a p.d.f?

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

If so determine the probability that the variate having this density will fall in the interval (1, 2).

Sol: we observe $f(x) \geq 0$.

also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^{\infty}$$

$$= -[0-1]$$

$$= 1$$

$\therefore f(x)$ is a p.d.f^n.

Scanned with CamScanner

Normal Distribution

The Contineous probability distribution having probability density function $f(x)$ given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as normal distribution.

evidently $f(x) \geq 0$ → we are showing it is a p.d.f

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

put $t = \frac{x-\mu}{\sqrt{2}\sigma}$ or $x = \mu + \sqrt{2}\sigma t$,

we have $dx = \sqrt{2}\sigma\,dt$

t also varies from $-\infty$ to ∞

hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\sigma\,dt$

$$= \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}\sqrt{\pi}} \cdot \cancel{\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2} \cdot \cancel{\sigma}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \times 2 \underline{\int_{0}^{\infty} e^{-t^2}\,dt}$$

But $\int_{0}^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$ by gamma functions

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{2}} = 1$$

both the conditions satisfies

∴ It is a probability density fn

Mean & Standard deviation of Normal distribution

Mean $(\mu) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \quad - ①$$

put $t^2 = \frac{(x-\mu)^2}{2\sigma^2}$

$t = \frac{(x-\mu)}{\sqrt{2}\sigma}$

$\sqrt{2}\sigma t = x-\mu$

$x = \mu + \sqrt{2}\sigma t$

D.w.r. to x

$dx = 0 + \sqrt{2}\sigma . dt$

$dx = \sqrt{2}\sigma\, dt$

$$① \Rightarrow \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-t^2} \sqrt{2}\cancel{\sigma}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t) e^{-t^2} . dt$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \mu e^{-t^2}\, dt + \sqrt{2}\sigma \int_{-\infty}^{\infty} t e^{-t^2}\, dt \right.$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

$$= \underset{\substack{\downarrow \\ \text{even function}}}{\frac{\mu}{\sqrt{\pi}} \times 2 \int_{0}^{\infty} e^{-t^2} dt} + \sigma \frac{\sqrt{2}}{\sqrt{\pi}} \underset{\substack{\downarrow \\ \text{odd function}}}{\int_{-\infty}^{\infty} t e^{-t^2} dt}$$

$$= \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2} dt + 0$$

$$= \frac{\cancel{2}\mu}{\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$\therefore$ mean $= \mu$

Variance (σ^2) $= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $t^2 = \frac{(x-\mu)^2}{2\sigma^2}$

$$t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$\sqrt{2}\sigma t = x-\mu$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma\, dt$$

t also vary from $-\infty$ to ∞

$$= \int_{-\infty}^{\infty} 2t^2\sigma^2 \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-t^2} \sqrt{2}\cancel{\sigma}\, dt$$

$$= 2\sigma^2 \int_{-\infty}^{\infty} t^2 \frac{1}{\sqrt{\pi}} e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

↳ even function

$$= \frac{2\sigma^2}{\sqrt{\pi}} \times 2 \int_{0}^{\infty} t^2 e^{-t^2} dt$$

$f(t) = t^2 e^{-t^2}$

$f(-t) = +f(t)$ ①

even function

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \times \int_{0}^{\infty} t\,(2t\,e^{-t^2})\, dt$$

$u = t \quad v = 2t\,e^{-t^2}$

W.K.T $\int uv\,dt = u\int v\,dt - \int\int v.dt.\,u'\,dt$

$$\int v.dt = \int 2t\,e^{-t^2} = \cancel{2t}\cdot\frac{e^{-t^2}}{-\cancel{2t}} = -e^{-t^2}$$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[t\,(-e^{-t^2}) \right]_0^{\infty} - \int_0^{\infty} -(e^{-t^2})\, 1.dt \right\}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left\{ 0 - 0 + \int_0^{\infty} e^{-t^2}.\,dt \right\}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

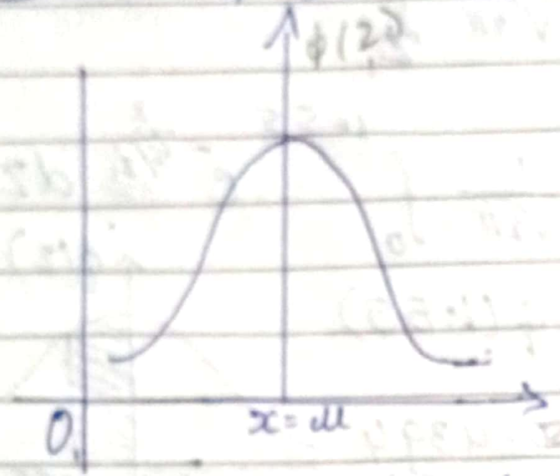
$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$= \sigma^2$$

Variance $= \sigma^2$

hence we can say that variance / S.D of normal distribution is equal to the variance / S.D of the given distribution

NOTE①: The graph of the probability function $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ & is called normal probability curve. The shape of the curve is



The line $x = \mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x = \mu$ is 0.5

NOTE②: $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} \, dz$

represents the area under standard normal curve from 0 to Z.

① $\int_{-\infty}^{\infty} \phi(z) \, dz = 1$

② $\int_{-\infty}^{0} \phi(z) \, dz = \int_0^{\infty} \phi(z) \, dz = 1/2$

③ $P(-\infty \leq z \leq \infty)$ or $P(z \geq 0) = 1/2$

also $P(-\infty < z < z_1) = P(-\infty < z \leq 0) + P(0 \leq z < z_1)$

④ $P(z < z_1) = 0.5 + \phi(z_1)$

⑤ also $P(z > z_2) = P(z \geq 0) - P(0 \leq z < z_2)$

i.e $P(z > z_2) = 0.5 - \phi(z_2)$

problems

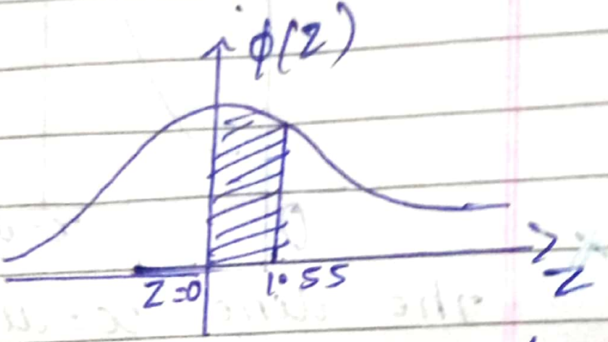
① Find the area under the standard normal curve b/w $Z=0$ & 1.55

Sol^{n}: $= \frac{1}{\sqrt{2\pi}}\int_{Z_1}^{Z_2} e^{-Z^2/2}\, dZ$

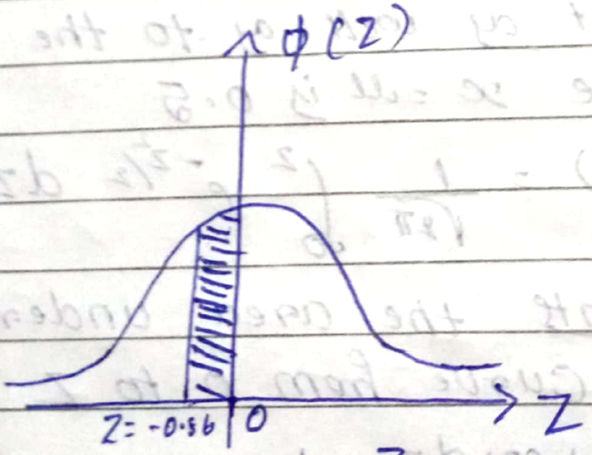
$= \frac{1}{\sqrt{2\pi}}\int_{0}^{1.55} e^{-Z^2/2}\, dZ$

$= \phi(1.55)$

$= 0.4394$



② Find the area under the standard normal curve between $Z=-0.86$ & $Z=0$



area $= \frac{1}{\sqrt{2\pi}}\int_{Z_1}^{Z_2} e^{-Z^2/2}\, dZ$

$= \frac{1}{\sqrt{2\pi}}\int_{-0.86}^{0} e^{-Z^2/2}\, dZ$

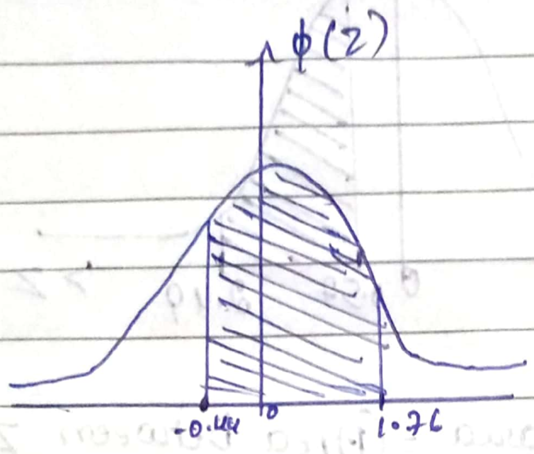
$= 1/\sqrt{2\pi}\int_{0}^{0.86} e^{-Z^2/2}\, dZ$

$= \phi(0.86)$

$(-0.86 \le z \le 0) = 0.30511$ //

③ find the area of Standard normal curve between $z = -0.44$ & $z = 1.76$

Sol^n:



$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{1.76} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-0.44}^{0} e^{-z^2/2}\, dz + \int_{0}^{1.76} e^{-z^2/2}\, dz \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{0.44} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{1.76} e^{-z^2/2}\, dz$$

$$= \phi(0.44) + \phi(1.76)$$

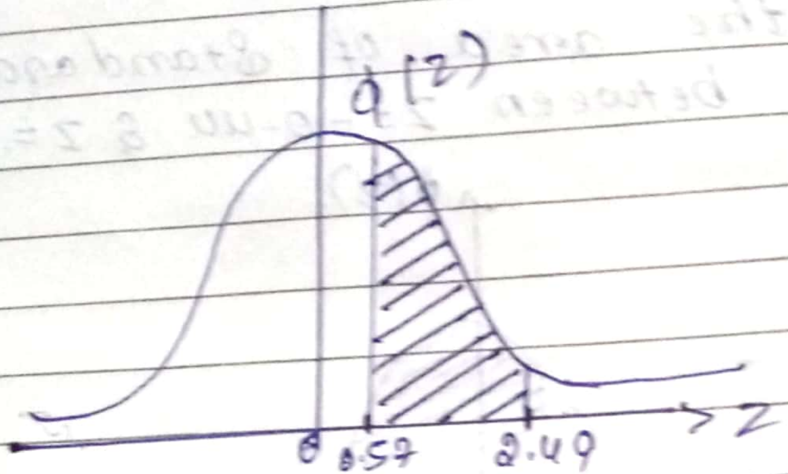
$$= 0.17003 + 0.4608$$

$$= \underline{\underline{0.63083}}$$

Do yourself

④ $Z = 0.57$ to 2.49

Solⁿ:

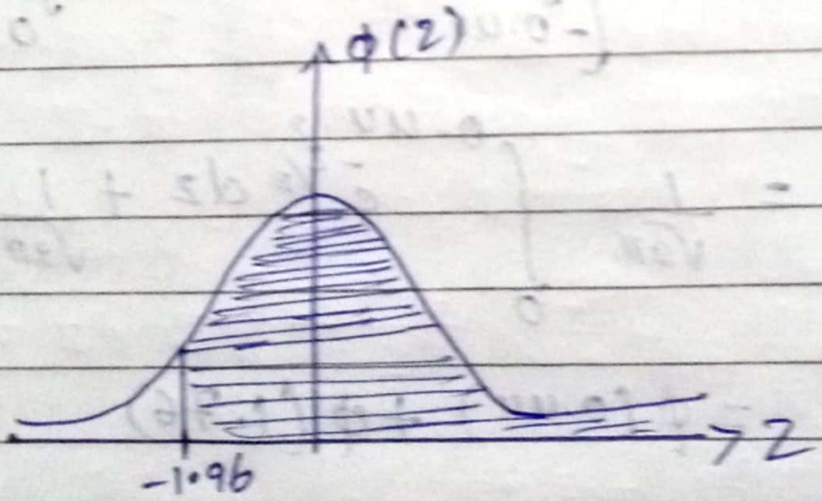


Required area = (Area between $Z=0$ to 2.49)
− (Area b/ω $Z=0$ to 0.57)

$= \phi(2.49) - \phi(0.57)$
$= 0.4936 - 0.2157$
$= 0.2779$ //

⑤ Find the area of Standard normal curve to right of $Z = -1.96$

Solⁿ:



Required Area = (Area between $Z=-1.96$ to 0)
+ (Area to right of $Z=0$)

⑨

$= (\text{Area b/w } z = 0 \text{ to } 1.96) + 0.5$ by symm-try

$= \phi(1.96) + 0.5$

$= 0.4750 + 0.5$

$= 0.9750$

$P(Z \geq -1.96) = \underline{\underline{0.9750}}$

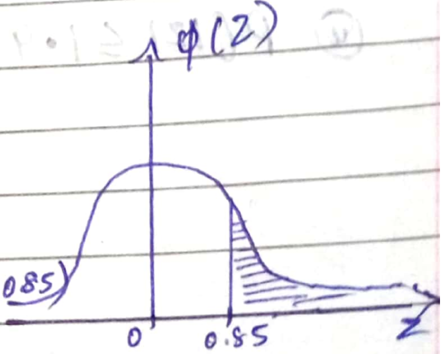
⑥ Evaluate the following probabilities with the help of normal probability tably.

① $P(Z \geq 0.85)$

② $P(-1.64 \leq Z \leq -0.88)$

③ $P(Z \leq -2.43)$

④ $P(|Z| \leq 1.94)$



Sol^{n}: ① $P(Z \geq 0.85)$

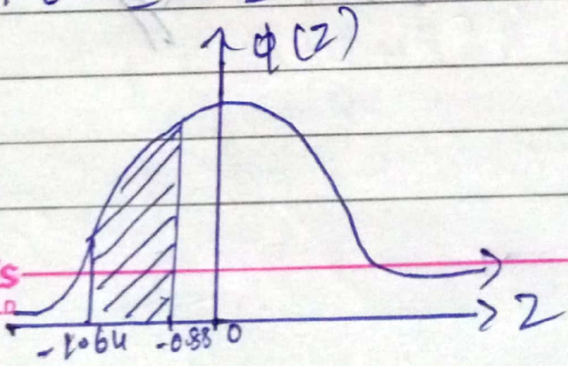
$P(Z \geq 0.85) = P(Z \geq 0) - P(0 \leq Z \leq 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.30234$

$= 0.1977$ //

② $P(-1.64 \leq Z \leq -0.88)$



$P(-1.64 \leq Z \leq -0.88) = P(0.88 \leq Z \leq 1.64)$

$= P(0 \leq Z \leq 1.64) - P(0 \leq Z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.31057$

$= 0.1389$ //

③ $P(Z \leq -2.43) = P(Z \geq 2.43)$

$= P(Z \geq 0) - P(Z \leq 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$= 0.0075$ //

④ $P(|Z| \leq 1.94) = P(-1.94 \leq Z \leq 1.94)$

$= P(-1.94 \leq Z \leq 0) + P(0 \leq Z \leq 1.94)$

$= P(0 \leq Z \leq 1.94) + P(0 \leq Z \leq 1.94)$

$= \phi(1.94) + \phi(1.94)$

$= 2\phi(1.94)$

$= 2 \times 0.47381$

$= 0.94762$ //

⑧ If x is a normal variate with mean 30 & standard deviation 5 find the probability that
① $26 \leq x \leq 40$ ② $x \geq 45$

Sol: we have standard normal variate

$$Z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$$

① To find $P(26 \leq x \leq 40)$

If $x = 26$ $Z = -0.8$; If $x = 40$, $Z = 2$

$\therefore P(-0.8 \leq Z \leq 2) = P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 2)$

$= P(0 \leq Z \leq 0.8) + P(0 \leq Z \leq 2)$

$= \phi(0.8) + \phi(2)$

$= 0.28814 + 0.47725$

$= 0.76539$

② $P(x \geq 45)$

If $x = 45$, $Z = 3$ hence we have to find $P(Z \geq 3)$

$P(Z \geq 3) = P(Z \geq 0) - P(Z \leq 3)$
or $P(0 \leq Z \leq 3)$

$= 0.5 - \phi(3)$

$= 0.5 - 0.4987$

$= 0.0013$

Doyourself

① If x is normally distributed with mean 18 & S.D 4, find the following
① $P(x \geq 20)$ ② $P(x \leq 20)$

④ The marks of 1000 students in an examination follows a normal distribution with mean 70 & S.D 5. Find the no. of students whose marks will be
① Lessthan 65 ② morethan 75 ③ b/w 65 & 75

Soln: let x represents the marks of students

By data $\mu = 70$, $\sigma = 5$ $z = \frac{x-\mu}{\sigma}$

$$z = \frac{x-70}{5}$$

① If $x = 65$, $z = -1$

we have to find $P(z < -1)$

$$P(z<-1) = P(z>1)$$
$$= P(z \geq 0) - P(0<z<1)$$
$$= 0.5 - \phi(1)$$
$$= 0.5 - 0.3413$$
$$= 0.1587$$

No% of students scoring lessthan 65 marks

$$= 1000 \times 0.1587$$
$$= 158.7$$
$$\approx 159$$

(ii) If $x = 75$, $z = 1$, we have to find $P(z > 1)$

$$P(z > 1) = P(z \geq 0) - P(0 < z < 1)$$
$$= 0.5 - \phi(1)$$
$$= 0.5 - 0.3413$$
$$= 0.1587$$

∴ No/. of Students scoring morethan 75 marks $= 1000 \times 0.1587 = 158.7 \approx 159$

(iii) we have to find $P(65 < x < 75)$

If $x = 65$ $z = -1$, If $x = 75$, $z = 1$

hence

$$P(-1 < z < 1) = P(-1 < z < 0) + P(0 < z \leq 1)$$
$$= P(0 < z < +1) + P(0 < z < 1)$$
$$= 2P(0 < z < 1)$$
$$= 2\phi(1)$$
$$= 2(0.3413)$$
$$= 0.6826$$

No/. of Students scoring marks b/w 65 & 75 $= 1000 \times 0.6826$
$= 682.6$
≈ 683

(2) In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an

average life of 2000 hrs & S.D of 60 hrs

If a firm purchases 2500 bulbs find the no. of bulbs that are likely to last for ① morethan 2100 hrs

② Lessthan 1950 hrs

③ b/w 1900 to 2100 hrs

Solⁿ: By data $\mu = 2000$, $\sigma = 60$.

S.N.V. $Z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

① To find $P(x > 2100)$

If $x = 2100$ $Z = \frac{2100-2000}{60} = \frac{100}{60} = 1.67$

$P(x > 2100) = P(Z > 1.67)$

$= P(Z \geq 0) - P(0 < Z < 1.67)$

$= 0.5 - \phi(1.67)$

$= 0.5 - 0.4525$

$= 0.0475$ //

No. of bulbs that are likely to last for morethan 2100 hrs is 2500×0.0475

$= 118.75 \approx 119$ //

② To find $P(x < 1950)$

If $x = 1950$, $Z = -5/6 = -0.83$

$P(x < 1950) = P(Z < -0.83)$

$= P(Z > 0.83)$

$= P(Z \geq 0) - P(0 < Z < 0.83)$

$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967$

$= \underline{0.2033}$

No/. of bulbs that are likely to last for less than 1950 hrs is 2500×0.2033

$= 508.25$

≈ 508//

③ To find $p(1900 < x < 2100)$

If $x = 1900$, $Z = -1.67$ & if $x = 2100$

$Z = 1.67$

$$p(1900 < x < 2100) = p(-1.67 < Z < 1.67)$$

$$= p(-1.67 < Z < 0) + p(0 < Z < 1.67)$$

$$= p(0 < Z < 1.67) + p(0 < Z < 1.67)$$

$$= \phi(1.67) + \phi(1.67)$$

$$= 2\phi(1.67)$$

$$= 2 \times 0.45254$$

$$= \underline{\underline{0.90508}}$$

No/. of bulbs that are likely to last between 1900 & 2100 hrs

$= 2500 \times 0.90508$

$= 2262.7$

$\approx \underline{\underline{2263}}$

MODULE-4 CURVE FITTING

Fitting of straight line $y = a + bx$ and $y = ax + b$

$y = a + bx$

$\Sigma y = na + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

$y = ax + b$

$\Sigma y = a\Sigma x + nb$

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

Fitting of parabola $y = a + bx + cx^2$ and $y = ax^2 + bx + c$

$y = a + bx + cx^2$

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

$y = ax^2 + bx + c$

$\Sigma y = a\Sigma x^2 + b\Sigma x + nc$

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$

$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

Fitting of curves of the form $y = ab^x$, $y = ax^b$, $y = ae^{bx}$

$y = ab^x$

log on BS

$\log_e y = \log_e(ab^x)$

$\log_e y = \log_e a + \log_e b^x$

$\log_e y = \log_e a + x\log_e b$

$Y = A + BX \rightarrow ①$

$Y = \log_e y \quad A = \log_e a$

$B = \log_e b \quad X = x$

$\Sigma Y = nA + B\Sigma X \rightarrow ②$

$\Sigma XY = A\Sigma X + B\Sigma X^2 \rightarrow ③$

$\log_e a = A$ & $a = e^A$

$\log_e b = B$ & $b = e^B$

$y = ab^x$

$y = ae^{bx}$

log on BS

$\log_e y = \log_e ae^{bx}$

$\log_e y = \log_e a + \log_e e^{bx}$

$\log_e y = \log_e a + bx\log_e e$

$\log_e y = \log_e a + bx$

$Y = A + bx$

$Y = \log_e y \quad A = \log_e a$ & $a = e^A$

$\Sigma Y = nA + b\Sigma X$

$\Sigma xY = A\Sigma x + b\Sigma x^2$

$$y = ax^b$$

log on BS

$$\log_e y = \log_e ax^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + bX$$

$$Y = \log_e y \quad A = \log_e a \quad X = \log_e x$$

$$\Sigma Y = nA + b\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2$$

---

co-efficient of corelation and equation of lines of regression

co-relation co-efficient $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{xy}^2}{2\sigma_x \sigma_y}$

$$\bar{x} = \frac{\Sigma x}{n} \qquad \bar{y} = \frac{\Sigma y}{n} \qquad \bar{z} = \frac{\Sigma z}{n}$$ where $z = x - y$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 \qquad \sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 \qquad \sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2$$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{xy}^2}{2\sigma_x \sigma_y}$$

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

Regression and co-efficient of correlation

Find $\bar{x}$ $\bar{y}$

$X = x - \bar{x}$, $Y = y - \bar{y}$, XY, X^2, Y^2

$$Y = \frac{\Sigma XY}{\Sigma X^2} \cdot X$$

$$r = \pm\sqrt{(\text{co-eff of } x)(\text{co-eff of } y)}$$

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}}$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} \cdot Y$$

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$

In this topic we discuss the method of finding a specific relation $y = f(x)$ for the data to satisfy as accurately as possible and such as equation is called the best fitting equation or the a curve of best fit.

The method is called as the method of least squares described as follows.

Suppose $y = f(x)$ is an approximate relation that fits into a given data (x_i, y_i) $i = 1, 2, 3 \ldots n$ then y_i's are called the observed values and $y_i = f(x_i)$ are called the expected values. Their difference $R_i = y_i - Y_i$ are called the residual or estimate errors.

Fitting of a straight line $y = a + bx$

consider a set of n given values (x, y) for fitting the straight line $y = a + bx$ where a and b are parameters to be determine. Normal equations for fitting the straight line $y = a + bx$

$$y = a + bx \rightarrow ①$$

$$\Sigma y = na + b\Sigma x \rightarrow ②$$

$$\sum_{1}^{n} a = a + a + a \ldots$$

$$= \underline{na}$$

x^{ly} x in ①

$$xy = xa + bx^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

$\therefore$ The normal equations of $y = a + bx$ are

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

NOTE :- Normal equations for $y = ax + b$ are

$$\Sigma y = a\Sigma x + nb$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x$$

Working procedure for problems

1) we first write the normal equations appropriate to curve of fit

2) we prepare the relevant table and find the value of summation present in normal equations we substitute these values to arrive at

a system of equations in unknown parameters

3) we find the parameters by solving and substitute in given equations

---

problems

1) Fit a straight line $y = a + bx$ in the least square sense for the data.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Normal equations for $y = a + bx$ are

$$\Sigma y = na + b\Sigma x \rightarrow ①$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow ②$$

$\boxed{n=8}$

x	y	xy	x^2
1	1	1	1
3	2	6	9
4	4	16	16
6	4	24	36
8	5	40	64
9	7	63	81
11	8	88	121
14	9	126	196
56	40	364	524

Normal equations of ① and ② becomes

$$40 = 8a + 56b$$

$$364 = 56a + 524b$$

$\boxed{a = 0.545}$ $\boxed{b = 0.636}$

put a and b in $y = a + bx$

$$\underline{\underline{y = 0.545 + 0.636x}}$$

---

2) Fit a straight line $y = a + bx$ for the data

x	0	1	2	3	4	5	6
y	2	1	3	2	4	3	5

Normal equations for $y = a + bx$ are

$$\Sigma y = na + b\Sigma x \rightarrow ①$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow ②$$

$\boxed{n=7}$

x	y	xy	x^2
0	2	0	0
1	1	1	1
2	3	6	4
3	2	6	9
4	4	16	16
5	3	15	25
6	5	30	36
21	20	74	91

Normal equations ① and ② becomes

$$20 = 7a + 21b$$

$$74 = 21a + 91b$$

$\boxed{a = 1.357}$ $\boxed{b = 0.5}$

put a and b in $y = a + bx$

$$y = 1.357 + 0.5x$$

3) Find the equation of best fitting straight line $y = ax + b$ for the following data

x	5	10	15	20	25
y	16	19	23	26	30

Normal equations for $y = ax + b$ are

$$\Sigma y = a\Sigma x + nb \rightarrow ①$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \rightarrow ②$$

$\boxed{n = 5}$

$\left\{ \sum_1^n b = nb \right.$

$y = ax + b \rightarrow ①$

$\Sigma y = a\Sigma x + nb$

x^{ly} x in ①

$xy = ax^2 + bx$

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

x	y	xy	x^2
5	16	80	25
10	19	190	100
15	23	345	225
20	26	520	400
25	30	750	625
75	114	1885	1375

Normal equations ① and ② becomes

$$114 = 75a + 5b$$

$$1885 = 1375a + 75b$$

$\boxed{a = 0.7}$ $\boxed{b = 12.3}$

put a and b in $y = ax + b$

$$y = 0.7x + 12.3$$

4) Find the equation of the best fitting straight line for the data.

x	0	1	2	3	4	5
y	9	8	24	28	26	20

we shall fit the straight line $y = ax + b$ for the given data

The normal equations are

$$\Sigma y = a\Sigma x + nb \rightarrow ①$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \rightarrow ② \qquad \boxed{n=6}$$

x	y	xy	x^2
0	9	0	0
1	8	8	1
2	24	48	4
3	28	84	9
4	26	104	16
5	20	100	25
15	115	344	55

Normal equations ① and ② becomes

$$115 = 15a + 6b$$

$$344 = 55a + 15b$$

$$\boxed{a = 3.228} \qquad \boxed{b = 11.09}$$

put a and b in $y = ax + b$

$$\underline{y = 3.228x + 11.09}$$

5) fit a straight line for the data

x	50	70	100	120
y	12	15	21	25

we shall fit the straight line $y = ax + b$ for the given data.

The normal equations are

$$\Sigma y = a\Sigma x + nb \rightarrow ①$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \rightarrow ② \qquad \boxed{n=4}$$

x	y	xy	x^2
50	12	600	2500
70	15	1050	4900
100	21	2100	10000
120	25	3000	14400
340	73	6750	31800

Normal equations ① and ② becomes

$$73 = 340a + 4b$$

$$6700 = 31800a + 340b$$

$$\boxed{a = 0.188} \qquad \boxed{b = 2.23}$$

put a and b in $y = ax + b$

$$\underline{y = 0.188x + 2.23}$$

6) A simply supported beam carries a concentrated load p at its midpoint corresponding to various values of p the maximum deflection y is measured and is given in the following table

p	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

Find the law of the form $y = a + bp$ and hence estimate y when $p = 150$

$y = a + bp$

Normal equations are

$\Sigma y = na + b\Sigma p \rightarrow ①$

$\Sigma py = a\Sigma p + b\Sigma p^2 \rightarrow ②$

$\boxed{n = 6}$

p	y	py	p^2
100	0.45	45	10000
120	0.55	66	14400
140	0.60	84	19600
160	0.70	112	25600
180	0.80	144	32400
200	0.85	170	40000
900	3.95	621	142000

Normal equations ① and ① becomes

$3.95 = 6a + 900b$

$621 = 900a + 142000b$

$\boxed{a = 0.048}$ $\boxed{b = 0.004}$

put a and b in $y = a + bp$

$y = 0.048 + 0.004p$

when $p = 150$

$\boxed{y = 0.65}$

Fitting of a second degree parabola $y = a + bx + cx^2$

consider a set of n given values (x, y) for fitting the curve of $y = a + bx + cx^2$ where a, b and c are parameters to be determined

$y = a + bx + cx^2 \rightarrow ①$

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

x^{ly} x in eq^n ①

$xy = xa + bx^2 + cx^3 \rightarrow ②$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \rightarrow ③$

x^{ly} x in eq^n ②

$x^2y = ax^2 + bx^3 + cx^4$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \rightarrow ④$

Normal equations are

$na + b\Sigma x + c\Sigma x^2 = \Sigma y$

$a\Sigma x + b\Sigma x^2 + c\Sigma x^3 = \Sigma xy$

$a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 = \Sigma x^2 y$

Problems

1) Fit a parabola of second degree $y = a + bx + cx^2$ for the data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	2.3

Normal equations for $y = a + bx + cx^2$ are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$$\boxed{n = 5}$$

x	y	xy	x^2	x^2y	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1	1.8	1	1
2	1.3	2.6	4	5.2	8	16
3	2.5	7.5	9	22.5	27	81
4	2.3	9.2	16	36.8	64	256
10	8.9	21.1	30	66.3	100	354

Normal equations become

$$5a + 10b + 30c = 8.9$$

$$10a + 30b + 100c = 21.1$$

$$30a + 100b + 354c = 66.3$$

$\boxed{a = 1.07}$ $\boxed{b = 0.415}$ $\boxed{c = -0.021}$

put a, b and c in $y = a + bx + cx^2$

$$y = 1.07 + 0.415x - 0.021x^2$$

---

2) Fit a parabola $y = a + bx + cx^2$ by the method of least squares for the data.

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

x	y	x^2	x^3	x^4	xy	x^2y
2	3.07	4	8	16	6.14	12.28
4	12.85	16	64	256	51.4	205.6
6	31.47	36	216	1296	188.82	1132.9
8	57.38	64	512	4096	459.04	3672.3
10	91.29	100	1000	10000	912.9	9129
30	196.06	220	1800	15664	1618.3	14152.1

Normal equations for $y = a + bx + cx^2$ are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$$\boxed{n = 5}$$

Normal equations become

$$5a + 30b + 220c = 196.06, \quad 30a + 220b + 1800c = 1618.3$$

$$220a + 1800b + 15664c = 14152.13$$

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$\boxed{a = 0.696}$ $\boxed{b = -0.85}$ $\boxed{c = 0.99}$

put a, b and c in $y = a + bx + cx^2$

$$y = a + bx + cx^2$$

$$\underline{y = 0.696 - 0.85x + 0.99x^2}$$

---

3) Fit a second degree parabola $y = A + Bx + Cx^2$ in the least square sense for the following data and hence find y at $x = 6$.

x	1	2	3	4	5
y	10	12	13	16	19

The normal equations of $y = A + Bx + Cx^2$ are

$$\Sigma y = nA + B\Sigma x + C\Sigma x^2$$

$$\Sigma xy = A\Sigma x + B\Sigma x^2 + C\Sigma x^3$$

$$\Sigma x^2 y = A\Sigma x^2 + B\Sigma x^3 + C\Sigma x^4$$

$\boxed{n = 5}$

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	13	9	27	81	39	117
4	16	16	64	256	64	256
5	19	25	125	625	95	475
15	70	55	225	979	232	906

Normal equations become

$$5A + 15B + 55C = 70$$

$$15A + 55B + 225C = 232$$

$$55A + 225B + 979C = 906$$

$\boxed{A = 9.4}$ $\boxed{B = 0.48}$ $\boxed{C = 0.28}$

put A, B and C in $y = A + Bx + Cx^2$

$$y = 9.4 + 0.48x + 0.28x^2$$

when $x = 6$

$$y = 9.4 + 0.48 \times 6 + 0.28 \times 6^2$$

$$\boxed{y = 22.36}$$

---

4) Fit a second degree parabola $y = Ax^2 + Bx + C$ in the least square sense for the following data and hence find y at $x = 6$

x	1	2	3	4	5
y	10	12	13	16	19

Normal equations of $y = Ax^2 + Bx + C \rightarrow$ ① are

$\Sigma y = A\Sigma x^2 + B\Sigma x + nC$

x^{ly} x in eq^n ①

$xy = Ax^3 + Bx^2 + Cx \rightarrow$ ②

$\Sigma xy = A\Sigma x^3 + B\Sigma x^2 + C\Sigma x$

x^{ly} x in eq^n ②

$x^2y = Ax^4 + Bx^3 + Cx^2 \rightarrow$ ③

$\Sigma x^2y = A\Sigma x^4 + B\Sigma x^3 + C\Sigma x^2$

∴ Normal equations are

$A\Sigma x^2 + B\Sigma x + nC = \Sigma y$

$A\Sigma x^3 + B\Sigma x^2 + C\Sigma x = \Sigma xy$ $\boxed{n=5}$

$A\Sigma x^4 + B\Sigma x^3 + C\Sigma x^2 = \Sigma x^2y$

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	13	9	27	81	39	117
4	16	16	64	256	64	256
5	19	25	125	625	95	475
15	70	55	225	979	232	906

Normal equations becomes

$55A + 15B + 5C = 70$

$225A + 55B + 15C = 232$

$979A + 225B + 55C = 906$

$\boxed{A = 0.28}$ $\boxed{B = 0.48}$ $\boxed{C = 9.4}$

put A, B and C in $y = Ax^2 + Bx + C$

$y = 0.28x^2 + 0.48x + 9.4$

when $x = 6$

$y = 0.28 \times 36 + 0.48 \times 6 + 9.4$

$y = \underline{22.36}$

---

5) Fit a parabola for the data

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

we shall fit the parabola $y = a + bx + cx^2$ for the given data. The normal equations are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$\boxed{n=9}$

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	6	4	8	16	12	24
3	7	9	27	81	21	63
4	8	16	64	256	32	128
5	10	25	125	625	50	250
6	11	36	216	1296	66	396
7	11	49	343	2401	77	539
8	10	64	512	4096	80	640
9	9	81	729	6561	81	729
45	74	285	2025	15333	421	2771

Normal equations becomes

$$9a + 45b + 285c = 74$$
$$45a + 285b + 2025c = 421$$
$$285a + 2025b + 15333c = 2771$$

$\boxed{a = -0.928}$ $\boxed{b = 3.52}$ $\boxed{c = -0.26}$

put a, b and c in $y = a + bx + cx^2$

$$y = -0.928 + 3.52x + [-0.26x^2]$$
$$\underline{y = -0.928 + 3.52x - 0.26x^2}$$

6) Find the best values of a, b, c if the equation $y = a + bx + cx^2$ is to fit most closely to the following observations

x	-2	-1	0	1	2
y	-3.15	-1.39	0.62	2.88	5.378

Normal equations of $y = a + bx + cx^2$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

$\boxed{n=5}$

x	y	x^2	x^3	x^4	xy	x^2y
-2	-3.15	4	-8	16	6.3	-12.6
-1	-1.39	1	-1	1	1.39	-1.39
0	0.62	0	0	0	0	0
1	2.88	1	1	1	2.88	2.88
2	5.378	4	8	16	10.756	21.512
0	4.338	10	0	34	21.326	10.402

Normal equations becomes

$$5a + 0b + 10c = 4.338$$
$$0a + 10b + 0c = 21.326$$
$$10a + 0b + 34c = 10.402$$

$\boxed{a = 0.621}$ $\boxed{b = 2.132}$ $\boxed{c = 0.123}$

put a, b and c in $y = a + bx + cx^2$

$$\underline{y = 0.621 + 2.132x + 0.123x^2}$$

Fitting of curves of the form $y = ab^x$, $y = ax^b$, $y = ae^{bx}$

Consider $y = ab^x$

Take log on both sides [to base e]

$$\log_e y = \log_e (ab^x)$$

$$\log_e y = \log_e a + \log_e b^x$$

$$\log_e y = \log_e a + x \log_e b$$

$$Y = A + BX \rightarrow ①$$

where $Y = \log_e y$ $A = \log_e a$ $B = \log_e b$ $X = x$

Normal equations becomes

$$\Sigma Y = nA + B\Sigma X \rightarrow ②$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \rightarrow ③$$

Solving ② and ③ we obtain A and B we have

$$\log_e a = A \quad \log_e b = B$$

$$a = e^A \quad b = e^B$$

Substitute the values of a and b in $y = ab^x$

---

problems

1) Fit a curve of the form $y = ab^x$ in the least square sense for the following data

x	0	2	4	5	7	10
y	100	120	256	390	710	1600

Consider $y = ab^x$

apply Log on both sides

$$\log_e y = \log_e (ab^x)$$

$$\log_e y = \log_e a + \log_e b^x$$

$$\log_e y = \log_e a + x \log_e b$$

$$Y = A + BX$$

where $Y = \log_e y$

$A = \log_e a$ & $a = e^A$ $B = \log_e b$ & $b = e^B$ $X = x$

Normal equations of $Y = A + BX$

$$\Sigma Y = nA + B\Sigma X$$

$$\boxed{n=6}$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

$x = X$	y	$Y = \log_e y$	X^2	XY
0	100	4.6052	0	0
2	120	4.7875	4	9.575
4	256	5.5452	16	22.1808
5	390	5.9661	25	29.830
7	710	6.5653	49	45.957
10	1600	7.3778	100	73.778
28		34.8471	194	181.321

Normal equations becomes

$$6A + 28B = 34.8471$$

$$28A + 194B = 181.3214$$

By solving we get

$$\boxed{A = 4.4298} \quad \boxed{B = 0.2953}$$

But $a = e^A = e^{4.4298}$ $\quad a = \underline{83.914}$

$b = e^B = e^{0.2953}$ $\quad b = \underline{1.343}$

Then required curve is $y = \underline{(83.914)(1.3435)^x}$

---

2) Fit a curve of the form $y = ab^x$ for the data hence find the estimation for y when $x = 8$

x	1	2	3	4	5	6	7
y	87	97	113	129	202	195	193

consider $y = ab^x$

Apply log on both sides

$$\log_e y = \log_e(ab^x)$$

$$\log_e y = \log_e a + \log_e b^x$$

$$\log_e y = \log_e a + x \log_e b$$

$$Y = A + BX$$

where $Y = \log_e y \quad A = \log_e a \quad B = \log_e b \quad X = x$

Normal equation of $Y = A + BX$ are

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

$$\boxed{n=7}$$

$x = X$	y	$Y = \log_e y$	XY	X^2
1	87	4.47	4.47	1
2	97	4.57	9.14	4
3	113	4.73	14.19	9
4	129	4.86	19.44	16
5	202	5.31	26.55	25
6	195	5.27	31.62	36
7	193	5.26	36.82	49
28		34.47	142.23	140

Normal equations becomes

$$7A + 28B = 34.47$$

$$28A + 140B = 142.23$$

$$\boxed{A = 4.303} \quad \boxed{B = 0.1554}$$

$A = \log_e a$

$a = e^A$

$a = e^{4.303}$

$a = \underline{\underline{73.92}}$

$B = \log_e b$

$b = e^B$

$b = e^{0.1554}$

$b = \underline{\underline{1.1681}}$

put a and b in $y = ab^x$

$$y = (73.92)(1.1681)^x$$

when $x = 8$

$$y = (73.92)(1.1681)^8$$

$$\boxed{y = 256.18}$$

---

3) At constant temperature the pressure p and volume v of a gas are connected by relation pv^γ = constant. Find the best fitting equation of this form to the following data and estimate v when p=4

P(kg.sqcm)	0.5	1.0	1.5	2.0	2.5	3.0
v (c.c)	1620	1000	750	620	520	460

consider $pv^\gamma = k$ where k is constant take log on BS

$$\log_e pv^\gamma = \log_e k$$

$$\log_e p + \gamma \log_e v = \log_e k$$

$$\log_e p = \log_e k - \gamma \log_e v$$

let us take $\log_e p = y$ $\quad \log_e v = x$

$\log_e k = a$ $\quad -\gamma = b$

so that we have

$y = a + bx$ which is a straight line

The associated normal equations are

$\Sigma y = na + b\Sigma x \rightarrow$ ①

$\Sigma xy = a\Sigma x + b\Sigma x^2 \rightarrow$ ②

$\boxed{n = 6}$

p	v	$x = \log_e v$	$y = \log_e p$	xy	x^2
0.5	1620	7.39	-0.69	-5.099	54.612
1.0	1000	6.91	0	0	47.748
1.5	750	6.62	0.40	2.648	43.824
2.0	620	6.43	0.69	4.436	41.344
2.5	520	6.25	0.92	5.75	39.062
3.0	460	6.13	1.1	6.743	37.576
		39.73	2.42	14.478	264.163

Normal equations ① and ② Becomes

$6a + 39.73b = 2.42$

$39.73a + 264.1689b = 14.47$

$\boxed{a = 9.7934}$ $\boxed{b = -1.42}$

But $\log_e K = a \Rightarrow K = e^a$

$\boxed{K = 17915}$

$-\gamma = b \Rightarrow b = 1.42$

The curve of fit $pv^{\gamma} = k$

$pv^{1.42} = 17915$

when $p = 4$, $4v^{1.42} = 17915$

$v^{1.42} = \frac{17915}{4}$ & $v^{1.42} = 4478$

$v = (4478)^{1/1.42} = 372.59 \simeq 373$

when $\boxed{p = 4}$ $\boxed{v = 373}$

---

4) Fit a curve of the form $y = ae^{bx}$ for the data

x	0	2	4
y	8.12	10	31.82

consider $y = ae^{bx}$

Apply log on both sides

$\log_e y = \log_e [ae^{bx}]$

$\log_e y = \log_e a + bx \log_e e$

$\boxed{\log_e e = 1}$

$\log_e y = \log_e a + bx$

$Y = A + bx$

where $Y = \log_e y$ $A = \log_e a \Rightarrow a = e^A$

Normal equations are

x	y	$Y = \log_e y$	xY	x^2
0	8.12	2.09	0	0
2	10	2.30	4.60	4
4	31.82	3.46	13.8	16
6		7.85	18.4	20

$$\Sigma Y = nA + b\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2 \qquad \boxed{n=3}$$

Normal equations Becomes

$$3A + 6B = 7.85$$

$$6A + 20B = 18.44$$

$$\boxed{A = 1.932} \qquad \boxed{B = 0.3425}$$

But $A = \log_e a \Rightarrow a = e^A = e^{1.932} = \underline{6.9033}$

put a and b in $y = ae^{bx}$

$$\underline{y = 6.9033\, e^{0.3425x}}$$

---

5) Find the equation of the best fitting curve in the form $y = ae^{bx}$ for the data

x	5	6	7	8	9	10
y	133	55	23	7	2	2

consider $y = ae^{bx}$

apply Log on both sides

$$\log_e y = \log_e [ae^{bx}] \qquad \boxed{\log_e e = 1}$$

$$\log_e y = \log_e a + bx \log_e e$$

$$\log_e y = \log_e a + bx$$

$$Y = A + bx$$

where $Y = \log_e y$

$$A = \log_e a \Rightarrow a = e^A$$

Normal equations Becomes

$$\Sigma Y = nA + b\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2$$

$$\boxed{n=6}$$

x	y	$Y = \log_e y$	xy	x^2
5	133	4.89	24.45	25
6	55	4.007	24.042	36
7	23	3.135	21.945	49
8	7	1.946	15.568	64
9	2	0.693	6.237	81
10	2	0.693	6.93	100
45		15.364	99.172	355

Normal equations becomes

$$6A + 45b = 15.364$$

$$45A + 355b = 99.172$$

$$\boxed{A = 9.443} \quad \boxed{b = -0.92}$$

$$A = \log_e a \quad \& \quad a = e^A \quad \& \quad e^{9.443}$$

$$\underline{a = 12619}$$

put a and b in $y = ae^{bx}$

$$\underline{y = 12619\, e^{-0.92x}}$$

---

6) Fit a least square geometric curve $y = ax^b$ from the following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

consider $y = ax^b$

Apply log on Both sides

$$\log_e y = \log_e ax^b$$

$$\log_e y = \log_e a + \log_e x^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + BX$$

where $Y = \log_e y \quad A = \log_e a \quad X = \log_e x$

Normal equations are

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \boxed{n=5}$$

$$5A + 4.787b = 6.1092$$

$$4.787A + 6.1993b = 9.0804$$

$$\boxed{A = -0.693} \quad \boxed{b = 2}$$

x	y	$X = \log_e x$	$Y = \log_e y$	X^2	XY
1	0.5	0	-0.6931	0	0
2	2	0.6931	0.6931	0.4804	0.4804
3	4.5	1.0986	1.5041	1.2069	1.6524
4	8	1.3863	2.0794	1.9218	2.8827
5	12.5	1.6094	2.5257	2.5902	4.0649
		4.7874	6.1092	6.1993	9.0804

$A = \log_e a \Rightarrow a = e^A$

$a = e^{-0.693}$ $\boxed{a = 0.5}$ $\boxed{b = 2}$

put a and b in $y = ax^b$

$y = (0.5)x^2$

Correlation :- co-variation of two independent magnitudes is known as co-relation.

co-relation co-efficient :- The numerical measure of corelation between two variables x and y is known as pearsons co-efficient of correlation usually denoted by r.

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{xy}^2}{2\sigma_x \sigma_y}$$

Regression :- It is an estimation of 1 independent variable in terms of the order.

The best fitting straight line of the form $y = ax+b$ is called regression line of y on x and $x = ay + b$ is called regression line of x on y

---

Working procedure to find the co-efficient of corelation and equation of lines of regression

1) Prepare the table showing the columns x, y, z and x^2, y^2, z^2 and finding the summation of x, y, z, x^2, y^2, z^2

2) Find $\bar{x} = \frac{\Sigma x}{n}$, $\bar{y} = \frac{\Sigma y}{n}$, $\bar{z} = \frac{\Sigma z}{n}$ where $z = x - y$

3) Find $\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2$, $\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2$ and

$$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2$$

4) $$r = \frac{a_x^2 + a_y^2 - a_z^2}{2 a_x a_y}$$

5) Find the lines of regression

$$y - \bar{y} = r \frac{a_y}{a_x}(x - \bar{x})$$

$$x - \bar{x} = r \frac{a_x}{a_y}(y - \bar{y})$$

NOTE : Find the lines of regression and co-efficient of corelation

1) Find $\bar{x}, \bar{y}$

2) prepare the table showing $X = x - \bar{x}$, $Y = y - \bar{y}$, XY, X^2, Y^2

3) Find ΣXY, ΣX^2, ΣY^2

4) Find regression lines i,e in the form of

a) $$Y = \frac{\Sigma XY}{\Sigma X^2} \cdot X$$

$$y - \bar{y} = \frac{\Sigma XY}{\Sigma X^2}(x - \bar{x})$$ regression line y on x

b) $$X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

$$x - \bar{x} = \frac{\Sigma XY}{\Sigma Y^2}(y - \bar{y})$$ regression line x on y

5) Find the corelation co-efficient

$$r = \pm \sqrt{(\text{co-Eff of } x)(\text{co-Eff of } y)}$$

co-relation co-efficient can also be written as

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}}$$

NOTE : co-efficient of co-relation numerically does not exceed unity i,e $-1 \le r \le 1$

Problems

1) Compute the co-efficient of correlation and the equation of the lines of regression for the data

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

$\boxed{n = 7}$

x	y	$z = x-y$	x^2	y^2	z^2
1	9	-8	1	81	64
2	8	-6	4	64	36
3	10	-7	9	100	49
4	12	-8	16	144	64
5	11	-6	25	121	36
6	13	-7	36	169	49
7	14	-7	49	196	49
28	77	-49	140	875	347

$$\bar{x} = \frac{\Sigma x}{n} = \frac{28}{7} = 4$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{77}{7} = 11$$

$$\bar{z} = \frac{\Sigma z}{n} = \frac{-49}{7} = -7$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{140}{7} - (4)^2 = 20 - 16 = 4$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{875}{7} - (11)^2 = 125 - 121 = 4$$

$$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2 = \frac{347}{7} - (-7)^2 = 49.57 - 49 = 0.57$$

Substitute all these in

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x \sigma_y} = \frac{4 + 4 - 0.57}{2 \times \sqrt{4} \times \sqrt{4}} = \frac{7.43}{8} = 0.928 \simeq 0.93$$

$$\boxed{r = 0.93}$$

The lines of regression are given by

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 11 = \frac{0.93 \times \cancel{2}}{\cancel{2}}(x - 4)$$

$$y - 11 = 0.93x - 3.72$$

$$y = 0.93x - 3.72 + 11$$

$$\underline{y = 0.93x + 7.28}$$

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 4 = \frac{(0.93)\cancel{(2)}}{\cancel{(2)}}(y - 11)$$

$$x - 4 = 0.93y - 10.23$$

$$x = 0.93y - 10.23 + 4$$

$$\underline{x = 0.93y - 6.23}$$

These are lines of regression

---

2) Find the correlation co-efficient for two groups and equation of lines of regression

A	92	89	87	86	83	77	71	63	53	50
B	86	83	91	87	68	85	52	82	37	57

$\boxed{n = 10}$

$A = x$	$B = y$	$z = x - y$	x^2	y^2	z^2
92	86	6	8464	7396	36
89	83	6	7921	6889	36
87	91	-4	7569	8281	16
86	77	9	7396	5921	81
83	68	15	6889	4624	225
77	85	-8	5929	7225	64
71	52	19	5041	2704	361
63	82	-19	3969	6724	361
53	37	16	2809	1369	256
50	57	-7	2500	3249	49
751	718	33	58487	54390	1485

$$\bar{x} = \frac{\Sigma x}{n} = \frac{751}{10} = 75.1$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{718}{10} = 71.8$$

$$\bar{z} = \frac{\Sigma z}{n} = \frac{33}{10} = 3.3$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{58487}{10} - (75.1)^2$$

$$\sigma_x^2 = 208.69 \quad \sigma_x = 14.446$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{54390}{10} - (71.8)^2$$

$$\sigma_y^2 = 283.76 \quad \sigma_y = 16.845$$

$$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2$$

$$\sigma_z^2 = \frac{1485}{10} - (3.3)^2$$

$$\sigma_z^2 = 148.5 - 10.89$$

$$\sigma_z^2 = 137.61$$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x \sigma_y} = \frac{208.69 + 283.76 - 137.61}{2 \times 14.446 \times 16.845}$$

$$r = 0.729$$

$$\boxed{r = 0.73}$$

The lines of regression are given by

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 71.8 = \frac{0.73 \times 16.845}{14.446}(x - 75.1)$$

$$y - 71.8 = 0.85(x - 75.1)$$

$$y - 71.8 = 0.85x - 63.84$$

$$y = 0.85x - 63.84 + 71.8$$

$$y = 0.85x + 7.965$$

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 75.1 = \frac{0.73 \times 14.446}{16.845}(y - 71.8)$$

$$x - 75.1 = 0.63(y - 71.8)$$

$$x - 75.1 = 0.63y - 45.234$$

$$x = 0.63y - 45.234 + 75.1$$

$$x = 0.63y + 29.866$$

---

3) Find the correlation co-efficient and equation of lines of a regression for the following values of x and y

x	1	2	3	4	5
y	2	5	3	8	7

$$\boxed{n = 5}$$

x	y	$z=x-y$	x^2	y^2	z^2
1	2	-1	1	4	1
2	5	-3	4	25	9
3	3	0	9	9	0
4	8	-4	16	64	16
5	7	-2	25	49	4
15	25	-10	55	151	30

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15}{5} = 3$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{25}{5} = 5$$

$$\bar{z} = \frac{\Sigma z}{n} = \frac{-10}{5} = -2$$

$$a_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{55}{5} - (3)^2 = 11 - 9 = 2 \text{ \& } \sqrt{2}$$

$$a_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{151}{5} - (5)^2 = 30.2 - 25 = 5.2$$

and $\sqrt{5.2}$

$$a_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2 = \frac{30}{5} - (-2)^2 = 6 - 4 = 2$$

$$r = \frac{a_x^2 + a_y^2 - a_z^2}{2 a_x a_y} = \frac{2 + 5.2 - 2}{2 \times \sqrt{2} \times \sqrt{5.2}} = 0.806 \quad \boxed{r = 0.81}$$

The lines of regression are given by

$$y - \bar{y} = r \frac{a_y}{a_x}(x - \bar{x})$$

$$y - 5 = \frac{0.81 \times \sqrt{5.2}}{\sqrt{2}}(x - 3)$$

$$y - 5 = 1.306(x - 3)$$

$$y - 5 = 1.306x - 3.918$$

$$y = 1.306x - 3.918 + 5$$

$$\underline{y = 1.306x + 1.082}$$

$$x - \bar{x} = r \frac{a_x}{a_y}(y - \bar{y})$$

$$x - 3 = \frac{0.81 \times \sqrt{2}}{\sqrt{5.2}}(y - 5)$$

$$x - 3 = 0.502(y - 5)$$

$$x - 3 = 0.502y - 2.51$$

$$x = 0.502y - 2.51 + 3$$

$$\underline{x = 0.502y + 0.49}$$

---

4) Obtain the lines of regression and hence find the co-efficient of correlation of the data

x	1	3	4	2	5	8	9	10	13	15
y	8	6	10	8	12	16	16	10	32	32

$\boxed{n = 10}$

$$\bar{x} = \frac{\Sigma x}{n} \qquad \bar{y} = \frac{\Sigma y}{n}$$

$$\bar{x} = \frac{70}{10} \qquad \bar{y} = \frac{150}{10}$$

$$\bar{x} = 7 \qquad \underline{\bar{y} = 15}$$

x	y	$X = x-\bar{x}$	$Y = y-\bar{y}$	X^2	Y^2	XY
1	8	-6	-7	36	49	42
3	6	-4	-9	16	81	36
4	10	-3	-5	9	25	15
2	8	-5	-7	25	49	35
5	12	-2	-3	4	9	6
8	16	1	1	1	1	1
9	16	2	1	4	1	2
10	10	3	-5	9	25	-15
13	32	6	17	36	289	102
15	32	8	17	64	289	136
70	150			204	818	360

we have lines of regression in the form

$$Y = \frac{\Sigma XY}{\Sigma X^2} X$$

$$y - \bar{y} = \frac{360}{204}(x - \bar{x})$$

$$y - 15 = 1.764(x - 7)$$

$$y = 1.764x - 12.348 + 15$$

$$y = 1.764x + 2.652$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

$$x - \bar{x} = \frac{360}{818}(y - \bar{y})$$

$$x - 7 = 0.44(y - 15)$$

$$x - 7 = 0.44y - 6.6$$

$$x = 0.44y + 0.4$$

$y = 1.764x + 2.652$ and

$x = 0.44y + 0.4$ are lines of regression.

co-efficient of corelation is

$$r = \sqrt{[\text{co-eff of } x][\text{co-eff of } y]}$$

$$r = \sqrt{1.764 \times 0.44}$$

$$\boxed{r = 0.88}$$

sign of r is positive since both the regression co-efficients are positive

---

5) Find the corelation co-efficient between x and y for the following data. Also obtain the regression lines

x	1	2	3	4	5	6	7	8	9	10
y	10	12	16	28	25	36	41	49	40	50

$\boxed{n = 10}$

$$\bar{x} = \frac{\Sigma x}{n} \qquad \bar{y} = \frac{\Sigma y}{n} \qquad \bar{z} = \frac{\Sigma z}{n}$$

$$\bar{x} = \frac{55}{10} \qquad \bar{y} = \frac{307}{10} \qquad \bar{z} = -\frac{252}{10}$$

$$\bar{x} = 5.5 \qquad \bar{y} = 30.7 \qquad \bar{z} = -25.2$$

x	y	z = x − y	x^2	y^2	z^2
1	10	−9	1	100	81
2	12	−10	4	144	100
3	16	−13	9	256	169
4	28	−24	16	784	576
5	25	−20	25	625	400
6	36	−30	36	1296	900
7	41	−34	49	1681	1156
8	49	−41	64	2401	1681
9	40	−31	81	1600	961
10	50	−40	100	2500	1600
55	307	−252	385	11387	7624

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{385}{10} - (5.5)^2$$

$$\sigma_x^2 = 38.5 - 30.25 = 8.25$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{11387}{10} - (30.7)^2$$

$$\sigma_y^2 = 1138.7 - 942.49 = 196.21$$

$$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2 = \frac{7624}{10} - (-25.2)^2$$

$$\sigma_z^2 = 762.4 - 635.04 = 127.36$$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x\sigma_y} = \frac{8.25 \times 196.21 \times 127.3}{2 \times \sqrt{8.25} \times \sqrt{196.21}}$$

$$\boxed{r = 0.958}$$

The lines of regression are given by

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 30.7 = \frac{0.958 \times \sqrt{196.21}}{\sqrt{8.25}}(x - 5.5)$$

$$y - 30.7 = 0.958 \times 4.876\,(x - 5.5)$$

$$y - 30.7 = 4.671\,(x - 5.5)$$

$$y = 4.671x - 25.691 + 30.7$$

$$y = 4.671x + 5.009$$

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 5.5 = \frac{0.958 \times \sqrt{8.25}}{\sqrt{196.21}}(y - 30.7)$$

$$x - 5.5 = 0.1964\,(y - 30.7)$$

$$x = 0.1964y - 6.029 + 5.5$$

$$x = 0.1964y - 0.529$$

6) the following data lines gives the age of husband (x) and age of wife (y) in years. Form the two regression lines and calculate the age of husband corresponding to 16 years age of wife.

x	36	23	27	28	28	29	30	31	33	35
y	29	18	20	22	27	21	29	27	29	28

$$\boxed{n = 10}$$

x	y	$z = x-y$	x^2	y^2	z^2
36	29	7	1296	841	49
23	18	5	529	324	25
27	20	7	729	400	49
28	22	6	784	484	36
28	27	1	784	729	1
29	21	8	841	441	64
30	29	1	900	841	1
31	27	4	961	729	16
33	29	4	1089	841	16
35	28	7	1225	784	49
300	250	50	9138	6414	306

$\bar{x} = \frac{\Sigma x}{n} = \frac{300}{10} = 30$

$\bar{y} = \frac{\Sigma y}{n} = \frac{250}{10} = 25$

$\bar{z} = \frac{\Sigma z}{n} = \frac{50}{10} = 5$

$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{9138}{10} - (30)^2$

$\sigma_x^2 = 913.8 - 900 = 13.8 = 3.71$

$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{6414}{10} - (25)^2$

$\sigma_y^2 = 641.4 - 625 = 16.4 = 4.04$

$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2 = \frac{306}{10} - (5)^2$

$\sigma_z^2 = 30.6 - 25 = 5.6 = 2.366$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x \sigma_y}$$

$$r = \frac{13.8 + 16.4 - 5.6}{2 \times 3.71 \times 4.04}$$

$\boxed{r = 0.82}$

The Regression Lines are given by

$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x})$

$y - 25 = \frac{0.82 \times 4.04}{3.71}(x - 30)$

$y - 25 = 0.8939(x - 30)$

$y = 0.8939x - 26.817 + 25$

$y = 0.8939x - 1.817$

$x - \bar{x} = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$

$x - 30 = \frac{0.82 \times 3.71}{4.04}(y - 25)$

$x - 30 = 0.7521(y - 25)$

$x = 0.7521y - 18.8049 + 30$

$x = 0.7521y + 11.19$

when $y = 16$, $x = ?$

$x = 0.7521y + 11.19$

$x = 0.7521 \times 16 + 11.19$

$x = 23.22 \simeq 23$

$\boxed{x = 23}$

Husbands age is 23 years corresponding to wife age of 16 years.

7) find the co-efficient of corelation for the following data.

x	10	14	18	22	26	30
y	18	12	24	6	30	36

$\boxed{n = 6}$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
10	18	−10	−3	100	9	30
14	12	−6	−9	36	81	54
18	24	−2	3	4	9	−6
22	6	2	−15	4	225	−30
26	30	6	9	36	81	54
30	36	10	15	100	225	150
120	126			280	630	252

$\bar{x} = \frac{\Sigma x}{n} = \frac{120}{6} = 20$

$\bar{y} = \frac{\Sigma y}{n} = \frac{126}{6} = 21$

$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2}\sqrt{\Sigma Y^2}} = \frac{252}{\sqrt{280}\sqrt{630}}$

$\boxed{r = 0.6}$

OR

x	y	$z = x - y$	x^2	y^2	z^2
10	18	−8	100	324	64
14	12	2	196	144	4
18	24	−6	324	576	36
22	6	16	484	36	256
26	30	−4	676	900	16
30	36	−6	900	1296	36
120	126	−6	2680	3276	412

$\bar{x} = \frac{\Sigma x}{n} = \frac{120}{6} = 20$

$\bar{y} = \frac{\Sigma y}{n} = \frac{126}{6} = 21$

$\bar{z} = \frac{\Sigma z}{n} = \frac{-6}{6} = 1$

$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{2680}{6} - (20)^2$

$\sigma_x^2 = 46.66 = 6.83$

$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{3276}{6} - (21)^2$

$\sigma_y^2 = 105 = 10.24$

$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2$

$\sigma_z^2 = \frac{412}{6} - (1)^2 = 67.66$

$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x\sigma_y} = \frac{46.66 + 105 - 67.66}{2 \times 6.83 \times 10.24}$

$\boxed{r = 0.6}$

---

8) obtain the lines of regression and hence find the co-efficient of corelation for the data.

x	10	14	18	22	26	30
y	18	12	24	6	30	36

$\boxed{n = 6}$

x	y	$X = x-\bar{x}$	$Y = y-\bar{y}$	X^2	Y^2	XY
10	18	-10	-3	100	9	30
14	12	-6	-9	36	81	54
18	24	-2	3	4	9	-6
22	6	2	-15	4	225	-30
26	30	6	9	36	81	54
30	36	10	15	100	225	150
120	126			280	630	252

$$\bar{x} = \frac{\Sigma x}{n} = \frac{120}{6} = 20$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{126}{6} = 21$$

we have lines of regression

$$Y = \frac{\Sigma XY}{\Sigma X^2} X$$

$$y-\bar{y} = \frac{252}{280}(x-\bar{x})$$

$$y-21 = 0.9(x-20)$$

$$y-21 = 0.9x - 18$$

$$y = 0.9x - 18 + 21$$

$$\underline{y = 0.9x + 3}$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

$$x-\bar{x} = \frac{252}{630}(y-\bar{y})$$

$$x-20 = 0.4(y-21)$$

$$x = 0.4y - 8.4 + 20$$

$$\boxed{x = 0.4y + 11.6}$$

$y = 0.9x + 3$ and $x = 0.4y + 11.6$ are lines of regression

co-relation co-efficient is

$$r = \sqrt{(\text{co-eff of } x)(\text{co-eff of } y)} = \sqrt{0.9 \times 0.4}$$

$$\boxed{r = 0.6}$$

$r = +0.6$ sign of r is positive since both regression co-efficients are positive

---

9) Find the correlation co-efficient by obtaining the lines of regression for the above data

x	17	18	19	19	20	20	21	22	21	23
y	12	16	14	11	15	19	22	15	16	20

$\boxed{n = 10}$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{200}{10} = 20$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{160}{10} = 16$$

x	y	$X = x-\bar{x}$	$Y = y-\bar{y}$	X^2	Y^2	XY
17	12	-3	-4	9	16	12
18	16	-2	0	4	0	0
19	14	-1	-2	1	4	2
19	11	-1	-5	1	25	5
20	15	0	-1	0	1	0
20	19	0	3	0	9	0
21	22	1	6	1	36	6
22	15	2	-1	4	1	-2
21	16	1	0	1	0	0
23	20	3	4	9	16	12
200	160			30	108	35

$$Y = \frac{\Sigma XY}{\Sigma X^2} X$$

$$y-\bar{y} = \frac{35}{30}(x-\bar{x})$$

$$y-16 = 1.16(x-20)$$

$$y-16 = 1.16x - 23.33$$

$$\underline{y = 1.16x - 7.33}$$

$$X = \frac{\Sigma XY}{\Sigma Y^2} Y$$

$$x-\bar{x} = \frac{35}{108}(y-\bar{y})$$

$$x-20 = 0.32(y-16)$$

$$x = 0.32y - 5.12 + 20$$

$$\underline{x = 0.32y + 14.88}$$

$y = 1.16x - 7.33$ and $x = 0.32y + 14.88$ are lines of regression co-relation co-efficient is

$$r = \sqrt{[\text{co-eff } x][\text{co-eff } y]}$$

$$r = \sqrt{(1.16)(0.32)}$$

$$\boxed{r = 0.609}$$

$r = +0.609$ sign r is positive since both regression co-efficients are positive

10) Given

	x-series	y-series
Mean	18	100
SD	14	20

and $r = 0.8$ write down the equation of lines of regression and hence find the most probable value of y where $x = 70$

By data $\bar{x} = 18$ and $\bar{y} = 100$

$\sigma_x = 14$ and $\sigma_y = 20$

we have the equation of regression lines are given by

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 100 = \frac{0.8 \times 20}{14}(x - 18)$$

$$\underline{\underline{y = 1.14x + 79.48}}$$

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 18 = \frac{0.8 \times 14}{20}(y - 100)$$

$$\underline{\underline{x = 0.56y - 38}}$$

when $x = 70$ we obtain from first equation

$$y = 1.14x + 79.48$$

$$y = 1.14 \times 70 + 79.48$$

$$\boxed{y = 159.28}$$

---

11) Show that if θ is the angle between the lines of regression then $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left[\frac{1 - r^2}{r}\right]$

WKT if θ is acute the angle between the lines $y = m_1 x_1 + c_1$ and $y = m_2 x_2 + c_2$ is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \rightarrow ⊛$$

we have lines of regression

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x}) \rightarrow ① \text{ and}$$

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$y - \bar{y} = \frac{\sigma_y}{r\sigma_x}(x - \bar{x}) \rightarrow ②$$

slopes of ① and ② are respectively given by

$$m_1 = r\frac{\sigma_y}{\sigma_x} \quad m_2 = \frac{\sigma_y}{r\sigma_x}$$

substitute these in the formula for $\tan\theta$ ⊛ we have

$$\tan\theta = \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1 + r\frac{\sigma_y}{\sigma_x}\frac{\sigma_y}{r\sigma_x}} = \frac{\frac{\sigma_y}{\sigma_x}\left[\frac{1}{r} - r\right]}{1 + \sigma_y^2/\sigma_x^2}$$

$$\tan\theta = \frac{\frac{\sigma_y}{\cancel{\sigma_x}}\left[\frac{1-r^2}{r}\right]}{\frac{\sigma_x^2+\sigma_y^2}{\cancel{\sigma_x^2}}}$$

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left[\frac{1-r^2}{r}\right]$$

---

12) In a bivarite distribution $\sigma_x = \sigma_y$ and the angle between the regression lines is $\tan^{-1}(3)$ find the correlation co-efficient

If θ is angle between the lines of regression we have

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left[\frac{1-r^2}{r}\right] \rightarrow ①$$

By data $\theta = \tan^{-1}(3)$ or $\tan\theta = 3$ and $\sigma_x = \sigma_y$

eq^n ① $\Rightarrow$ $3 = \frac{\sigma_x^{\cancel{2}}}{2\cancel{\sigma_x}}\left[\frac{1-r^2}{r}\right]$

$$3 = \frac{1-r^2}{2r}$$

$$6r = 1-r^2$$

$$r^2+6r-1=0$$

$$r = \frac{-6\pm\sqrt{40}}{2} = r = \frac{-6\pm 2\sqrt{10}}{2}$$

$$r = -3\pm\sqrt{10}$$

$r = -3 \pm \sqrt{10} = 0.1623$

$r = -3 - \sqrt{10} = -6.1623$

consider $\boxed{r = 0.1623}$ since r doesnot exceed unity

---

13) If $x = 4y+5$ and $y = Kx+4$ are the regression lines of x on y and y on x respectively. prove that $0 \le K \le 1/4$

WKT co-efficient of corelation

$$r = \sqrt{[\text{co-eff of } y][\text{co-eff of } x]}$$

$r = \sqrt{4(K)}$

$r = \sqrt{4K}$ squaring on both sides

$r^2 = 4K$

WKT $0 \leq r^2 \leq 1$

$0 \leq 4K \leq 1$

$0 \leq K \leq 1/4$

---

14) $8x - 10y + 66 = 0$ and $40x - 18y = 214$ are the two regression lines. Find the mean of x's, y's and the corelation co-efficient find σ_y if $\sigma_x = 3$

WKT regression lines passes through $\bar{x}$ and $\bar{y}$

$8\bar{x} - 10\bar{y} + 66 = 0$ $\Rightarrow$ $8\bar{x} - 10\bar{y} = -66$

$40\bar{x} - 18\bar{y} - 214 = 0$ $\quad 40\bar{x} - 18\bar{y} = 214$

By solving we get $\bar{x} = 13$ and $\bar{y} = 17$ we shall now rewrite the equation of the regression lines to find the regression co-efficients

$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ and $x - \bar{x} = r\frac{\sigma_y}{\sigma_x}(y - \bar{y})$

$10y = 8x + 66$ or $y = 0.8x + 6.6 \rightarrow ①$

$40x = 18y + 214$ $\quad x = 0.45y + 5.35 \rightarrow ②$

From ① $r\frac{\sigma_y}{\sigma_x} = 0.8$; $r\frac{\sigma_x}{\sigma_y} = 0.45$

co-relation co-efficient $r = \sqrt{0.8 \times 0.45}$ $\boxed{r = 0.6}$

Thus $r = 0.6$ since both the regression co-efficients are positive also $\sigma_x = 3$ by data and we have

$r\frac{\sigma_y}{\sigma_x} = 0.8 \Rightarrow r\frac{\sigma_y}{3} = 0.8 \Rightarrow 0.6\sigma_y = 2.4$ $\boxed{\sigma_y = 4}$

$r\frac{\sigma_x}{\sigma_y} = 0.45 \Rightarrow \frac{0.6 \times 3}{0.45} = \sigma_y$ $\boxed{\sigma_y = 4}$

Module-05

Sampling Theory and Joint probability distributions

Joint probability and Joint probability distribution

If X and Y are two discrete random variables, we define the joint probability function of X and Y by

$$P(X=x, Y=y) = f(x, y)$$

where $f(x,y)$ satisfy the conditions

$$f(x,y) \geq 0 \quad \text{and} \quad \sum_x \sum_y f(x,y) = 1$$

The second condition means that the sum of all values of x and y equal to one.

Joint probability table

X \ Y	y_1	y_2	...	y_n	Sum
x_1	J_{11}	J_{12}	...	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	...	J_{2n}	$f(x_2)$
...	...	...	...	...	...
x_m	J_{m1}	J_{m2}	...	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	...	$g(y_n)$	1

Marginal probability distribution

In the joint probability table, $f(x_1)$, $f(x_2)$, ... $f(x_m)$ respectively represents the sum of all the entries in the first row, second row ... m^{th} row. $g(y_1)$, $g(y_2)$, ... $g(y_n)$ respectively represents the sum of all the entries in the 1^{st} column, 2^{nd} ..... n^{th} column.

$$f(x_1) = J_{11} + J_{12} + \cdots + J_{1n} \quad g(y_1) = J_{11} + J_{21} + \cdots + J_{m1}$$

$$f(x_2) = J_{21} + J_{22} + \cdots + J_{2n}, \quad g(y_2) = J_{12} + J_{22} + \cdots + J_{m2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f(x_m) = J_{m1} + J_{m2} + \cdots + J_{mn}, \quad g(y_n) = J_{1n} + J_{2n} + \cdots + J_{mn}$$

$\{f(x_1), f(x_2) \ldots f(x_m)\}$ & $\{g(y_1), g(y_2) \ldots g(y_n)\}$ are called marginal probability distribution of X & Y respectively.

NOTE: $f(x_1) + f(x_2) + \cdots + f(x_m) = 1$

$g(y_1) + g(y_2) + \cdots + g(y_n) = 1$

Expectation: Expectation of X is denoted by $E(X)$ or μ_X

$$\mu_X = E(X) = \sum_{i=1}^{n} x_i f(x_i) \text{ or } \sum x f(x)$$

III'ly

$$\mu_Y = E(Y) = \sum_{i=1}^{n} y_i g(y_i) \text{ or } \sum y g(y)$$

$$E(XY) = \sum x_i y_j J_{ij}$$

Covariance:-

Co-variance of X & Y is denoted by

$+J_{m1}$

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y$$

$+J_{m2}$

..

Correlation:-

$+J_{mn}$

Correlation of X & Y is denoted by

~~cov (X (Y) & E(XY)~~

Not in Syllabus

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X^2 = E(X^2) - \mu_X^2$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$E(X^2) = \sum x_i^2 f(x_i)$$

$$E(Y^2) = \sum y_j^2 g(y_j)$$

NOTE: If X & Y are independent random variable then ① $E(XY) = E(X) \cdot E(Y)$

② $Cov(X,Y) = 0$ & hence $\rho(X,Y) = 0$

Problems

① The joint distribution of two random variables X & Y is as follows.

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute

① E(X) and E(Y)

② E(XY)

③ cov(X, Y)

④ σ_x^2, σ_y

Soln:

X \ Y	-4	2	7	sum
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
sum	3/8	3/8	1/4	1

} f(x)

g(y)

① $E(X) = \sum x_i f(x_i)$

$= 1 \times 1/2 + 5 \times 1/2$

$= 1/2 + 5/2$

$= 6/2$

$E(X) = 3$

② $E(Y) = \sum y_j g(y_j)$

$= -4 \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times 1/4$

$= -\frac{12}{8} + \frac{6}{8} + \frac{7}{4}$

$= \frac{-12 + 6 + 14}{8}$

$= \frac{8}{8} = 1 \Rightarrow E(Y) = 1$

They $\mu_x = E(X) = 3$

$\mu_y = E(Y) = 1$

② $E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-4)(1/8) + (1)(2)(1/4) + (1)(7)(1/8)$
$+ 5(-4)(1/4) + 5(2)(1/8) + 5(7)(1/8)$

$= -1/2 + 1/2 + 7/8 - 5 + 5/4 + 35/8$

$= \frac{42}{8} - \frac{15}{4}$

$= \frac{42-30}{8}$

$= \frac{12}{8}$

$E(XY) = \frac{3}{2}$ //

③ $\text{cov}(X,Y) = E(XY) - \mu_x \mu_y$

$= \frac{3}{2} - (3)(1)$

$7 \times 1/4$

$= \frac{3-6}{2}$

$= -3/2$ //

④ $\sigma_x^2 = E(x^2) - \mu_x^2$

$E(x^2) = \sum x_i^2 f(x_i)$

$= (1)^2(1/2) + 5^2(1/2)$

$= 1/2 + \frac{25}{2}$

$= 26/2$

$E(x^2) = 13$ //

$\sigma_x^2 = 13 - (3)^2 = 13 - 9 = 4$ //

$\sigma_x = \sqrt{4} \Rightarrow \sigma_x = 2$ //

~~E(~~ $\sigma_y^2 = E(y^2) - \mu_y^2$

$E(y^2) = \sum y_j^2 \cdot g(y_j)$

$E(y^2) = (-4)^2(3/8) + 2^2(3/8) + 7^2(1/4)$

$= 16 \times 3/8 + 4 \times 3/8 + 49 \times 1/4$

$= \frac{48 + 12 + 98}{8}$

$= \frac{\cancel{158}^{79}}{\cancel{8}_{4}}$

$$E(Y^2) = \frac{79}{4}$$

$$\sigma^2_{y} = \frac{79}{4} - (1)^2 = \frac{75}{4}$$

$$\sigma_y = \sqrt{75/4} = 4.33 //$$

② The Joint probability distribution table for two random variables X and Y is as follows.

X \ Y	−2	−1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distribution of X and Y also compute
① Expectation of X, Y and XY
② S.Ds of X, Y
③ Covariance of X and Y
Further verify that X and Y are dependent random variables

Solⁿ:

X \ Y	−2	−1	4	5	sum
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
Sum	0.3	0.3	0.1	0.3	1

last column & last row are marginal probability distributions.

① $E(X) = \sum x_i f(x_i)$

$= (1)(0.6) + 2(0.4)$

$= 0.6 + 0.8$

$E(X) = 1.4$ //

$E(X) = \mu_X = 1.4$ //

$E(Y) = \sum y_j g(y_j)$

$= -2(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3)$

$E(Y) = 1$

$E(Y) = \mu_Y = 1$ //

$E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0)$
$+ (1)(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1)$
$+ 2(4)(0.1) + 2(5)(0)$

$= 0.9$ //

② $\sigma^2_X = E(X^2) - \mu^2_X$ & $\sigma^2_Y = E(Y^2) - \mu^2_Y$

$E(X^2) = \sum x_i^2 f(x_i)$

$= (1)^2(0.6) + 2^2(0.4)$

$= 0.6 + 1.6$

$E(X^2) = 2.2$

$\sigma^2_X = E(X^2) - \mu^2_X$

$= 2.2 - (1.4)^2$

$= 0.24$

$\sigma_x = \sqrt{0.24}$

$\sigma_x = \underline{\underline{0.489}}$

$E(Y^2) = \sum y_j^2 g(y_j)$

$= (-2)^2(0.3) + (-1)^2(0.3) + 4^2(0.1) + 5^2(0.3)$

$= 4(0.3) + 0.3 + 16(0.1) + 25(0.3)$

$E(Y^2) = 10.6$

$\sigma^2 y = E(Y^2) - \mu^2 y$

$\sigma^2 y = 10.6 - 1^2$

$= 9.6$

$\sigma_y = \sqrt{9.6} = \underline{\underline{3.09}}$

$\sigma_x = \underline{\underline{0.489}}$ $\sigma_y = 3.09$//

③ $\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y)$

$= 0.9 - (1.4)(1)$

$= \underline{\underline{-0.5}}$

If x & y are independent random variable we must have $f(x_i)\,g(y_j) = J_{ij}$

Now $f(x_1)g(y_1) = (0.6)(0.3) = 0.18$

but $J_{11} = 0.1$

$0.18 \neq 0.1$

i.e $f(x_1)\,g(y_1) \neq J_{11}$

lll'ly for other values also condition is not satisfied.

hence we conclude that X & Y are dependent random variables.

③ Do yourself

The joint probability distribution table for two random variables X and Y is as follows

X \ Y	-2	5	8
1	0.21	0.35	0.14
2	0.09	0.15	0.06

determine the marginal probability distribution of X and Y, also find

① E(XY), E(X), E(Y)

② COV(X, Y)

④ Suppose x and y are independent random variable with the following respective distribution, find the joint distribution of x and y. Also verify that $cov(x,y)=0$

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Solⁿ:

X \ Y	$y_1=-2$	$y_2=5$	$y_3=8$	$f(x_i)$/sum
$x_1=1$	J_{11}	J_{12}	J_{13}	0.7
$x_2=2$	J_{21}	J_{22}	J_{23}	0.3
$g(y_j)$/sum	0.3	0.5	0.2	1

J_{ij} are obtained on multiplication of marginal entries

$J_{11}=(0.3)(0.7)=0.21$

$J_{12}=(0.7)(0.5)=0.35$

$J_{13}=(0.7)(0.2)=0.14$

$J_{21}=(0.3)(0.3)=0.09$

$J_{22}=(0.5)(0.3)=0.15$

$J_{23}=(0.3)(0.2)=0.06$

Joint distribution table is

X \ Y	-2	5	8	$f(x_i)$/sum
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$ or sum	0.3	0.5	0.2	1

$\mathrm{COV}(X,Y) = E(XY) - \mu_X \mu_Y$

$\mu_X = E(X) = \sum x_i f(x_i)$

$= (1)(0.7) + 2(0.3)$

$= \underline{\underline{1.3}}$

$\mu_Y = E(Y) = \sum y_j g(y_j)$

$= (-2)(0.3) + (5)(0.5) + 8(0.2)$

$= 3.5 //$

$E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-2)(0.21) + (1)(5)(0.35) + (1)(8)(0.14)$
$+ (2)(-2)(0.09) + (2)(5)(0.15) + (2)(8)(0.06)$

$= -0.42 + 1.75 + 1.12 - 0.36 + 1.5 + 0.96$

$= 4.55 //$

$\mathrm{cov}(X,Y) = 4.55 - (1.3)(3.5)$

$= 4.55 - 4.55$

$= \underline{\underline{0}}$

⑤ X and Y are independent variables X takes value 2, 5, 7 with probability $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$ repectively. Y takes values 3, 4, 5 with the probability $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$

ⓐ find the Joint probability distribution of X and Y.

Ⓑ S.T the Covariance of X and Y is equal to Zero

ⓒ find the probability distribution of $Z = X + Y$

Soln:

$X \backslash Y$	3	4	5	$f(x_i)$
2	J_{11}	J_{12}	J_{13}	$\frac{1}{2}$
5	J_{21}	J_{22}	J_{23}	$\frac{1}{4}$
7	J_{31}	J_{32}	J_{33}	$\frac{1}{4}$
$g(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

$J_{ij} = f(x_i)\, g(y_j)$

$J_{11} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$ $\quad J_{21} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$

$J_{12} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$ $\quad J_{22} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$

$J_{13} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$ $\quad J_{23} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$

$J_{31} = 1/12,\ J_{32} = 1/12,\ J_{33} = 1/12$

X \ Y	3	4	5	$f(x_i)$
2	1/6	1/6	1/6	1/2
5	1/12	1/12	1/12	1/4
7	1/12	1/12	1/12	1/4
$g(y_j)$	1/3	1/3	1/3	1

ⓑ $\text{cov}(XY) = E(XY) - \mu_X \mu_Y$

$$\mu_X = E(X) = \sum x_i f(x_i)$$

$$= 2(1/2) + 5(1/4) + 7(1/4)$$

$$= 1 + 5/4 + 7/4$$

$$= 1 + \frac{12}{4}$$

$$= 16/4 /\!/$$

$$\mu_X = 4 /\!/$$

$$\mu_Y = E(Y) = \sum y_j g(y_j)$$

$$\mu_Y = (1/3)(3) + 4(1/3) + 5(1/3)$$

$$= \frac{3+4+5}{3}$$

$$= 12/3 = 4 /\!/$$

$E(XY) = \sum x_i y_j J_{ij}$

$= (2)(3)(1/6) + (2)(4)(1/6) + (2)(5)(1/6) + 5(3)(1/12)$
$+ 5(4)(1/12) + (5)(5)(1/12) + 7(3)(1/12) +$
$7(4)(1/12) + 7(5)(1/12)$

$= 16$ //

$\mathrm{cov}(X, Y) = 16 - 4(4) = 16 - 16 = 0$ //

② $Z = X + Y$

let $Z_i = x_i + y_i$

$Z_i = \{5, 6, 7, 8, 9, 10, ~~10~~, 11, 12\}$

corresponding probabilities are

$1/6, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12$

probability distribution of $Z = X + Y$ is as follow

Z	5	6	7	8	9	10	11	12
$P(Z)$	$1/6$	$1/6$	$1/6$	$1/12$	$1/12$	$1/12 + 1/12 = 1/6$	$1/12$	$1/12$

$\sum P(Z) = \frac{3}{6} + \frac{\cancel{2}}{\cancel{12}\,6} + \frac{1}{6} + \frac{\cancel{2}}{\cancel{12}\,6}$

$= \frac{3+1+1+1}{6}$

$= \frac{6}{6}$

$= 1$

$5+5=10$
$7+3=10$
for both $1/12 + 1/12 = 2/12 = \frac{1}{6}$

(*) Given the following joint distribution of the random variables X & Y, find the correspanding marginal distribution also compute covariance & correlation of random variables X & Y.

X \ Y	1	3	9
2	$1/8$	$1/24$	$1/12$
4	$1/4$	$1/4$	0
6	$1/8$	$1/24$	$1/12$

Soln: marginal distribution of X & Y is

x_i	2	4	6
$f(x_i)$	$1/4$	$1/2$	$1/4$

y_j	1	3	9
$g(y_j)$	$1/2$	$1/3$	$1/6$

For understanding

$$f(x_1) = 1/8 + 1/24 + 1/12 = 12/48 = 1/4$$

$$f(x_2) = 1/4 + 1/4 + 0 = 2/4 = 1/2$$

$$f(x_3) = 1/8 + 1/24 + 1/12 = 1/4$$

$$g(y_1) = 1/8 + 1/4 + 1/8 = 2/8 + 1/4 = 2/4 = 1/2$$

$$g(y_2) = 1/24 + 1/4 + 1/24 = 2/24 + 1/4 = 1/12 + 1/4 = \frac{1+3}{12} = 1/3$$

$$g(y_3) = 1/12 + 0 + 1/12 = 2/12 = 1/6$$

$COV(X,Y) = E(XY) - \mu_x \mu_y$

$E(X) = \mu_x = \sum x_i f(x_i)$

$= (2)(1/4) + (4)(1/2) + (6)(1/4)$

$= 4$

$E(Y) = \mu_y = \sum y_j g(y_j)$

$= (1)(1/2) + 3(1/3) + 9(1/6)$

$= 3$

$E(XY) = \sum x_i y_j J_{ij}$

$= (2)(1)(1/8) + 2(3)(1/24) + 2(9)(1/12)$
$+ (4)(1)(1/4) + (4)(3)(1/4) + (4)(9)(0) + (6)(1)(1/8)$
$+ (6)(3)(1/24) + (6)(9)(1/12)$

$E(XY) = 12$

$Cov(X,Y) = 12 - 4(3)$

$= 0$

④ The Joint probability distribution of two discrete random variables X & Y is given by $f(x,y) = K(2x+y)$ where x & y are integers $\ni$:

$0 \le x \le 2,\ 0 \le y \le 3$

$= \frac{1+3}{12} = 1/3$

(a) Find K

(b) Find the marginal distributions of X & Y

(c) S.T the random variables X & Y are dependent

Sol^n:

$X = \{x_i\} = \{0, 1, 2\}$

$Y = \{y_j\} = \{0, 1, 2, 3\}$

$f(x, y) = K(2x + y)$ Joint probability distribution table is

X \ Y	0	1	2	3	Sum
0	0	K	2K	3K	6K
1	2K	3K	4K	5K	14K
2	4K	5K	6K	7K	22K
Sum	6K	9K	12K	15K	42K

(a) we must have $42K = 1$

$K = 1/42$

(b) marginal probability distribution is

x_i	0	1	2
$f(x_i)$	$6/42 = 1/7$	$14/42 = 1/3$	$22/42 = 11/21$

y_j	0	1	2	3
$g(y_j)$	$6/42 = 1/7$	$9/42 = 3/14$	$12/42 = 2/7$	$15/42 = 5/14$

(c) It can be easily seen that $f(x_i) g(y_j) f_{ij}$ hence random variables are dependent.

Stochastic process

Definitions :-

probability vector :

The vector $V = (v_1, v_2, v_3, \dots v_n)$ is called a probobility vector if each one of its components are non negative & their sum is equal to unity.

ex: $V = (1, 0)$

$V = (1/2, 1/2)$

$V = (1/4, 1/4, 1/2)$

are all probability vector

Stochastic matrix :- A square matrix P is said to be stochastic matrix if every row in the form of a probability vector.

exc: ① $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ② $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

Regular Stochastic matrix

A Stochastic matrix P is said to regular Stochastic matrix if all the entries of some power P^n are +ve ex: $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Consider $A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 3/4 \end{bmatrix}$

A is a regular stochastic matrix

Properties of Regular Stochastic matrix

① ⓐ P has a unique fixed point $x = (x_1, x_2 \cdots x_n) \ni; \; xP = x$

ⓑ P has unique fixed probobility vector $V = (V_1, V_2 \cdots V_n) \ni; \; VP = V$

where $V_i = \dfrac{x_i}{\sum_{i=1}^{n} x_i}$

ⓒ $P^2, P^3 \cdots$ approches the matrix V whose rows are fixed probability vector V.

ⓓ If u is any probobility vector then the sequence of vectors $uP, uP^2 \ldots uP^n$ approches the unique fixed probability vector V.

Problems:

① If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a Stochastic matrix and $V = [v_1, v_2]$ is a probability vector S.T VA is also a probability vector.

Sol^n: By data

$$a_1 + a_2 = 1, \quad b_1 + b_2 = 1, \quad v_1 + v_2 = 1$$

$$VA = [v_1 \ v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

$$VA = \begin{bmatrix} v_1 a_1 + v_2 b_1 & v_1 a_2 + v_2 b_2 \end{bmatrix}$$

we have to probability vector

So, $v_1 a_1 + v_2 b_1 + v_1 a_2 + v_2 b_2 = 1$

$$v_1 (a_1 + a_2) + v_2 (b_1 + b_2) = 1$$

$$v_1 (1) + v_2 (1) = 1$$

$$\underline{\underline{v_1 + v_2 = 1}}$$

$\therefore$ VA is a probability vector//

② Find the unique fixed probability vector of the regular Stochastic matrix $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Soln: we have to find $v = (x, y)$ where

$x + y = 1$ ∋; $VA = V$

$$[x, y]\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$$

$$\left[\frac{3}{4}x + \frac{y}{2}, \ \frac{x}{4} + \frac{y}{2}\right] = [x, y]$$

$$\frac{3}{4}x + \frac{y}{2} = x \quad - ①$$

$$\frac{3x}{4} + \frac{4}{2} = y \quad - ②$$

w.k.T $x + y = 1$

$y = 1 - x$

put $y = 1 - x$ in ①

$$\frac{3}{4}x + \frac{1-x}{2} = x$$

$$3x + 2(1-x) = 4x$$

$$3x + 2 - 2x = 4x$$

$$3x - 2x - 4x + 2 = 0$$

$$-3x + 2 = 0$$

$$-3x = -2$$

$$x = 2/3$$

$y = 1 - x = 1 - \frac{2}{3}$

$y = \frac{1}{3}$, $V = (x; y) = (2/3, 1/3)$

Thus $(2/3, 1/3)$ is the unique fixed probability vector.

③ Find the unique fixed probability vector for the regular Stochastic matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

Soln: we have to find $V = (x, y, z)$ where $x + y + z = 1$ ∋; $VA = V$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$\left[0 + y/6 + 0, \ x + y/2 + 2/3 z, \ 0 + y/3 + z/3\right] = [x, y, z$$

$y/6 = x$, $x + y/2 + 2/3 z = y$, $y/3 + z/3 = z$

$y = 6x$, $6x + 3y + 4z = 6y$, $y + z = 3z$

$y = 6x$, $6x - 3y + 4z = 0$, $y - 2z = 0$
$-①$ $-②$ $-③$

w.k.t $x+y+z=1$

[illegible]

$z=1-x-y$

w.k.t $y=6x$

$z=1-x-6x$

$z=1-7x$

③ becomes

$y-2z=0$

$y-2(1-7x)=0$

$6x-2+14x=0$

$20x-2=0$

$20x=2$

$\underline{\underline{x=1/10}}$

$y=6x=6\times 1/10$

$\underline{\underline{y=3/5}}$, $z=1-7x$

$=1-7\times 1/10$

$=1-7/10$

$z=\frac{3}{10}$ //

required unique fixed probability vector v is given by $v=(1/10, 3/5, 3/10)$

④ S.T $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular Stochastic matrix. Also find the associated unique fixed probability vector

Sol^n: consider $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P \cdot P^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

we observe that P^5 all the entries are +ve.

Thus P is a regular stochastic matrix

we have to find $v = (a, b, c)$ where $a+b+c=1$ ∋; $vP = v$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} a, & b, & c \end{bmatrix}$$

$$\left[0+0+\frac{c}{2}, \; a+0+\frac{c}{2}, \; 0+b+0\right] = [a, b, c]$$

$$\left[\frac{c}{2}, \; a+\frac{c}{2}, \; b\right] = [a, b, c]$$

$$\frac{c}{2} = a, \quad a+\frac{c}{2} = b, \quad b = c$$

$$c = 2a, \quad a+\frac{c}{2} = b, \quad b = c$$

— ①, — ②, — ③

$$b = c$$

$$b = 2a$$

use $b = 2a$ in ②

$$a+\frac{c}{2} = 2a$$

$$\frac{c}{2} = a$$

$$c = 2a$$

W.K.T $a+b+c=1$

$a+2a+2a=1$

$5a=1$

$a=\frac{1}{5}$

$\therefore\ b=\frac{2}{5},\quad c=2/5$

Thus $(1/5, 2/5, 2/5)$ is the required unique fixed probability vector of P.

⑤ Find the unique fixed probability vector of the regular stochastic matrix

$$P=\begin{bmatrix}0 & 1/2 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 1/2 & 1/2 & 0 & 0\\ 1/2 & 1/2 & 0 & 0\end{bmatrix}$$

Soln: we have to find $v=(a,b,c,d)$ where $a+b+c+d=1$ ∋; $vP=v$

$$\begin{bmatrix}a & b & c & d\end{bmatrix}\begin{bmatrix}0 & 1/2 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 1/2 & 1/2 & 0 & 0\\ 1/2 & 1/2 & 0 & 0\end{bmatrix}=\begin{bmatrix}a & b & c & d\end{bmatrix}$$

$$\left[0+\frac{b}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{2}+0+\frac{c}{2}+\frac{d}{2},\ \frac{a}{4}+\frac{b}{4}+0+0\right.$$
$$\left.a/4+b/4+0+0\right]=\begin{bmatrix}a & b & c & d\end{bmatrix}$$

$$\left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{4}+\frac{b}{4},\ \frac{a}{4}+\frac{b}{4}\right]=[a,\ b,\ c,\ d]$$

$$\frac{b}{2}+\frac{c}{2}+\frac{d}{2}=a \Rightarrow b+c+d=2a \quad \text{—①}$$

$$\frac{a}{2}+\frac{c}{2}+\frac{d}{2}=b \Rightarrow a+c+d=2b \quad \text{—②}$$

$$\frac{a}{4}+\frac{b}{4}=c \Rightarrow a+b=4c \quad \text{—③}$$

$$\frac{a}{4}+\frac{b}{4}=d \Rightarrow a+b=4d \quad \text{—④}$$

w.K.T $a+b+c+d=1$

$\underline{b+c+d}=1-a$

but use ① $2a=1-a$

$3a=1 \Rightarrow \underline{\underline{a=1/3}}$

$a+b+c+d=1$

use ② $\underline{a+c+d}=1-b$

$\hookrightarrow 2b=1-b$

$3b=1$

$\underline{\underline{b=1/3}}$

③ $\Rightarrow a+b=4c$

$1/3+1/3=4c$

$2/3=4c$

$c=\frac{2}{3\times 4} \Rightarrow \underline{\underline{c=1/6}}$

b, c, d]

$a+b=4d$

$\frac{1}{3}+\frac{1}{3}=4d$

$\frac{2}{3}=4d$

$d=\frac{1}{6}$

Thus $V=\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$ is the required unique fixed probability vector

Markov chains

A stochastic process which is ∋; the generation of the probability distribution depend only on the present state is called Markov process.

If the state space is discrete (finite or countably infinite) we say that the process is a discrete state process (or) chain. Then the markov process is known as a Markov chain.

It is defined as

① Each outcome belong to the finite set of the outcomes $\{a_1, a_2, \ldots a_m\}$

(ii) The outcome of any trial depend at most upon the outcome of the immediate preceeding trial.

probability P_{ij} is associated with every pair of sates (a_i, a_j) that a_j occours immediatly after a_i occuars. Such a stochastic process is called a finite markov chain

Transistion probability matrix

TPM of order m is denoted by P

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1} & P_{m2} & & P_{mm} \end{bmatrix}$$

elements of P have the following properties

① $0 \le P_{ij} \le 1$

② $\sum_{j=1}^{m} P_{ij} = 1$

$(i = 1, 2, \dots m)$

The above 2 propertiy satisfy the requirement of a stochastic matrix and hence we conclude that transition matrix of a markov chain is a stochastic matrix.

Higher transition probabilities

* The entry P_{ij} in the transition probability matrix p of the markov chain is the probability that system changes from state a_i to a_j in a single step i.e $a_i \to a_j$

* The probability that system changes from state a_i to a_j in exactly n steps is denoted by $P_{ij}^{(n)}$

* Let $p^{(n)} = [P_1^{(n)}, P_2^{(n)} \dots P_m^{(n)}]$ denote the nth step probability distribution at the end of n steps. Thus we have

$$p^{(1)} = p^{(0)}p, \quad p^{(2)} = p^{(1)}p = p^{(0)}.p.p = p^{(0)}p^2$$

$$\dots\dots \; p^{(n)} = p^{(0)}p^n$$

problems

① The transition matrix p of a markov chain is given by $\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with the initial probability distribution $p^{(0)} = (1/4, 3/4)$. Define and find the following

① $p_{21}^{(2)}$ ② $p_{12}^{(2)}$ ③ $p_1^{(2)}$

② the vector $p^{(0)}p^{(n)}$ approches

③ the matrix p^n approches

Sol^n; First we need to find P^2

$$P^2 = P \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

(i) $P_{21}^{(2)} = 9/16$ (ii) $P_{12}^{(2)} = 3/8$

(iii) $P^{(2)} = P^{(0)} P^{(2)}$

$$= [1/4, 3/4] \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

$$= \left[\frac{37}{64}, \frac{27}{64}\right] = [P_1^{(2)}, P_2^{(2)}]$$

(iv) $P_1^{(2)} = \frac{37}{64}$

i.e $P_1^{(2)}$ is the probability that the process is in the state a_1 after 2 steps

(v) The vector $P^{(0)} P^n$ approches the unique fixed probability vector P

let $V = (x, y)$ where $x + y = 1$

$$VP = V$$

$$[x, y]\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = [x, y]$$

$$\left[\frac{x}{2} + \frac{3y}{4}, \frac{x}{2} + \frac{y}{4}\right] = [x, y]$$

$$\frac{x}{2} + \frac{3y}{4} = x$$

$$2x + 3y = 4x$$

$$3y = 2x$$

but $x + y = 1$

$$y = 1 - x$$

$$3(1-x) = 2x$$

$$3 - 3x = 2x$$

$$3 = 5x$$

$$x = 3/5$$

$$x + y = 1 \Rightarrow y = 1 - x = 1 - 3/5 = 2/5$$

the vector $p^{(0)} p^{n}$ approchey the vector $(3/5, 2/5)$

(vi) p^{n} approchey the matrix $\begin{bmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{bmatrix}$

p^{n} approchey the matrix V whose rows are each the fixed probability vector p.

② The T.P.M of a markov chain is given by $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

and the initial probability distribution is $p^{(0)} = (1/2, 1/2, 0)$

Find $P_{13}^{(2)}$, $P_{23}^{(2)}$, $p^{(2)}$ and $P_1^{(2)}$

Soln: let us find two step transition matrix P^2

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$P_{13}^{(2)} = 3/8$ and $P_{23}^{(2)} = 1/2$

$$p(2) = p(0)\, p(2) = [1/2, 1/2, 0] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$p^{(2)} = [7/16, 1/8, 7/16]$$

$$P_1^{(2)} = 7/16$$

③ P.T the markov chain whose t.p.m is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector

Solⁿ: we shall S.T P is a regular Stochastic matrix

P^2

$$P^2 = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 7/12 \end{bmatrix}$$

Since all the entry P^2 are +ve we conclude that t.p.m P is regular hence it is irreducible.

we shall find the fixed probability vector of P.

If $V = (x, y, z)$ we shall find V ∋;

$VP = V$ where $x+y+z=1$

$$[x, y, z] \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \; y \; z]$$

$$\left[\frac{y}{2}+\frac{z}{2},\ \frac{2}{3}x+\frac{z}{2},\ \frac{x}{3}+\frac{y}{2}\right]=\left[x\ \ y\ \ z\right]$$

$$\frac{y}{2}+\frac{z}{2}=x$$

$2x = y+z \Rightarrow y+z = 2x \Rightarrow 2x-y-z=$

$\frac{2}{3}x+\frac{z}{2}=y \Rightarrow 4x+3z=6y \Rightarrow 4x-6y+$

$\frac{x}{3}+\frac{y}{2}=z \Rightarrow 2x+3y=6z$

$2x+3y-6z=0$ — ③

W.K.T $x+y+z=1$

$x = 1-y-z$

① ⇒ [illegible]

[illegible]

[illegible]

② ⇒ [illegible]

[illegible]

$① \Rightarrow 2(1-y-z)-y-z=0$

$2-2y-2z-y-z=0$

$2-3y-3z=0$

$-3y-3z+2=0$

$3y+3z-2=0$ — ④

~~② ⇒ $4(1-y-z) - 6y + 3z = 0$~~

~~$4 - 4y - 4z - 6y + 3z = 0$~~

~~$4 - 10y - z = 0$~~

~~$-10y - z + 4 = 0$ - ⑤~~

0 - ①

z=0 - ②

~~Solve ④ & ⑤~~

~~$3y + 3z - 2 = 0$~~

~~$(-)4y \underset{(-)}{\pm} z \underset{(+)}{\mp} = 0$~~

~~$-y + 0 = 0$~~

$\underline{y = 0}$

Solve ① & ②

$2x - y - z = 0$

$(-)\cancel{2x} \underset{(-)}{+} 3y \underset{(+)}{-} 6z = 0$

$-4y + 5z = 0$ - ⑤

Solve ④ & ⑤

$3y + 3z - 2 = 0 \times 4$

$-4y + 5z = 0 \times 3$

$\cancel{12y} + 12z - 8 = 0$

$-\cancel{12y} + 15z = 0$

$27z - 8 = 0$

$27z = 8$

$\underline{z = 8/27}$

put z in ⑤

$4y - 5\left(\frac{8}{27}\right) = 0$

$4y - \frac{40}{27} = 0$

$\cancel{4}y = \frac{\overset{10}{\cancel{40}}}{27}$

$y = \frac{10}{27}$

$x = 1 - y - z$

$= 1 - \frac{10}{27} - \frac{8}{27}$

$= \frac{27 - 18}{27}$

$= \frac{9}{27}$

$x = \frac{1}{3}$

Required stationary probability vector is $\left(\frac{1}{3}, \frac{10}{27}, \frac{8}{27}\right)$

③ A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on two Consecutive days. But if he visits club B on a particular day then the next day he is as likely to visit club B or club A. find transition matrix of this markov chain. also ⓐ S.T the matrix is a regular stochastic matrix & find the unique fixed probability vector.

ⓑ If the person had visited club B on monday, find the probability that he visits club A on Thursday.

Soln: The transition matrix P of markov chain is formulated as

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0 & 1 \\ B & 1/2 & 1/2 \end{array}$$

(i) $P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Since all the entries of P^2 are +ve

∴ P is a regular stochastic matrix

Now we shall find unique fixed probability vector

$$VP = V$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} 0 + \frac{y}{2}, & x + \frac{y}{2} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\frac{y}{2} = x \Rightarrow 2x = y$$

$$x + y/2 = y$$

$$2x + y = 2y \Rightarrow 2x - y = 0$$

w.r.t $x + y = 1$

$$y = 1 - x$$

$$2x + 1 + x = 0$$

$$3x + 1 = 0$$

$$3x = +1 \Rightarrow x = 1/3$$

$$y = 1 - 1/3$$

$$y = \frac{2}{3}$$

Thus $V = (1/3, 2/3)$

ⓑ let us suppose monday as day 1 then thursday will be 3 days after monday given that the person had visited club B on monday the probability that he visits club A after 3 days is equivalent to finding $a_{21}^{(3)}$ from P^3.

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$$

$a_{21}^{(3)} = 3/8$ required probability

* A student's study habits are as follows, if he studies one night, he is 70% sure not to study the next night, on the other hand if he doesnot study one night he is 60% sure not to study the next night. In the long run how often does he study?

Soln: A → Studying
B → not studying

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.3 & 0.7 \\ B & 0.4 & 0.6 \end{array}$$

we have to find unique fixed probabili vector $VP = V$

where $V = (x, y)$

w.r.t $x + y = 1$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} 0.3x + 0.4y & 0.7x + 0.6y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$0.3x + 0.4y = x$$
$$0.7x + 0.6y = y$$

$x+y=1$

$y=1-x$

$0.3x+0.4-0.4x=x$

$-0.1x-x=-0.4$

$+1.1x=+0.4$

$x=\frac{0.4}{1.1}$

$x=0.3636$ //

$y=1-4/11$

$=\frac{7}{11}$

$\underline{y=0.6363}$

Thus we can conclude that in the long run the student will study 4/11 of the time or 36.36 % of the time.

* The man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2 on the other hand if he smokes non filter cigarettes

one week there is a probability of 0.7 that he will smoke nonfilter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?

Sol^n: A: Smoking filter cigarettes
B: smoking non filter cigarettes
State space of system {A, B}
Associated transition matrix is

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.8 & 0.2 \\ B & 0.3 & 0.7 \end{array}$$

we have to find unique fixed probability vector $VP = V$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} 0.8x + 0.3y & 0.2x + 0.7y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$0.8x + 0.3y = x, \quad 0.2x + 0.7y = y$

$0.8x - x + 0.3y = 0, \quad 0.2x + 0.7y - y = 0$

$-0.2x + 0.3y = 0 \quad -①$ $\quad 0.2x - 0.3y = 0 \quad -②$

w.k.t $x + y = 1$

$y = 1 - x$

$① \Rightarrow 0.8x + 0.3 - 0.3x - x = 0$

$-0.5x = -0.3$

$x = \frac{0.3}{0.5}$

$x = 3/5$ or 0.6

$y = 1 - \frac{3}{5} = \frac{2}{5}$

(or) $y = 0.4$ //

$V = (x, y) = (3/5, 2/5) = (P_A, P_B)$

In the long run he will smoke filter cigarette $\frac{3}{5}$ (or) 60% of the time

* Three Boys A, B, C are throwing ball to each other A always throws the Ball to B & B always throws the Ball to C, C is just as likely to throw the ball to B as to A. If C was first person to throw the ball find the probability that after 3 throws.

(i) A has the ball

(ii) B has the ball

(iii) C has the ball

Soln: State space = {A B C}

t.p.m

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

Initially if C has ball

initial probability vector

$p^{(0)} = (0\ 0\ 1)$

Since the probability are desired after three rows we have to find

$p^3 = p^{(0)} \cdot p^3$

Refer problem no. ⑪ for P^3

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$p^{(3)} = p^{(0)} p^{(3)} = (1/4, 1/4, 1/2)$

After 3 throws the probability that ball is with A is 1/4, B is 1/4 and C is 1/2.

* Two boys B_1, B_2 & two Girls G_1, G_2 are throwing ball from one to other. Each boy throws the ball to the other boy with probability 1/2 and to each girl with probability 1/4. On otherhand each girl throw the ball

to each boy with probability $1/2$ and never to other girl. In the long run how often does each receive the ball.

Sol^n: State Space $= \{B_1, B_2, G_1, G_2\}$ and the associated t.p.m P is as follows

$$P = \begin{array}{c|cccc} & B_1 & B_2 & G_1 & G_2 \\ \hline B_1 & 0 & 1/2 & 1/4 & 1/4 \\ B_2 & 1/2 & 0 & 1/4 & 1/4 \\ G_1 & 1/2 & 1/2 & 0 & 0 \\ G_2 & 1/2 & 1/2 & 0 & 0 \end{array}$$

we need to find the fixed probability vector $V = (a, b, c, d)$

i; $VP = V$

$$(a\ b\ c\ d) \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a\ b\ c\ d]$$

$$\left[b/2 + c/2 + d/2,\ a/2 + c/2 + d/2,\ a/4 + b/4,\ a/4 + b/4\right] = [a\ b\ c\ d]$$

$$\frac{b}{2} + \frac{c}{2} + \frac{d}{2} = a, \quad \frac{a}{2} + \frac{c}{2} + \frac{d}{2} = b$$

$$b + c + d = 2a \qquad a + c + d = 2b$$

$$\text{— ①} \qquad \text{— ②}$$

$\frac{a}{4} + \frac{b}{4} = c$, $\quad a/4 + b/4 = d$

$a+b = 4c$ — ③ $\quad a+b = 4d$ — ④

W.K.T $\quad a+b+c+d = 1$

$b+c+d = 1-a$

^{lly} $a+c+d = 1-b$

① $\Rightarrow$ $1-a = 2a$

$3a = 1$

$\underline{\underline{a = 1/3}}$

② $\Rightarrow$ $1-b = 2b$

$3b = 1 \Rightarrow b = 1/3$

③ $\Rightarrow$ $1/3 + 1/3 = 4c \Rightarrow 4c = 2/3 \Rightarrow c = 1/6$

④ $\Rightarrow$ $1/3 + 1/3 = 4d \Rightarrow 4d = 2/3 \Rightarrow d = 1/6$

$V = (1/3, 1/3, 1/6, 1/6)$ is the required unique fixed probability vector

$a/4] = [a\ b\ c\ d]$